Electromagnetic-continuum-induced nonlinearity

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A nonrelativistic Hamiltonian describing interaction between a mechanical degree of freedom and radiation pressure is commonly used as an ultimate tool for studying system behavior in optomechanics. This Hamiltonian is derived from the equation of motion of a mechanical degree of freedom and the optical wave equation with time-varying boundary conditions. We show that this approach is deficient for studying higher-order nonlinear effects in an open resonant optomechanical system. Optomechanical interaction induces a large mechanical nonlinearity resulting from a strong dependence of the power of the light confined in the optical cavity on the mechanical degrees of freedom of the cavity due to coupling with electromagnetic continuum. This dissipative nonlinearity cannot be inferred from the standard Hamiltonian formalism.

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I. INTRODUCTION

Optomechanics attracted a lot of attention as a tool for transferring purely theoretical quantum mechanical notion to experimental labs [1]. Interaction of mechanical objects with light resulted in an efficient cooling of mechanical degrees of freedom [2–4] so single mechanical quanta became accessible. The light is able to manipulate the mechanical quanta, squeeze or entangle mechanical degrees of freedom [5–10]. Emission of coherent phonon radiation became possible [11].

Mechanical systems influence light as well, creating quantum entangled states between photons and phonons [12]. Quantum state transfer becomes possible between light and a mechanical system [13]. Finally, mechanical systems can modify the quantum properties of light, for instance, create squeezed light [14–16].

Nonlinear physics also benefited from the optomechanics [4,17,18]. High spectral purity optomechanical oscillators were created [19,20]. Efficient optical frequency harmonics arising from the stimulated Brillouin scattering were used to generate narrow-linewidth light [21] as well as low-noise radio-frequency signals [22]. Generation of a phonon frequency comb as well as mode locking of a mechanical distributed system were demonstrated [23].

The beauty of an optomechanical interaction is in its clear physical picture based on Maxwell equations. Optical wave impinging on a mechanical object transfers its momentum to the object. Both cavity frequency and photon number changes as the result of such an interaction. Intricate physical phenomena can occur in the system if the mechanical body is moving fast, if it absorbs or scatters light, if its size is comparable with the optical wavelength, etc. However, the system simplifies significantly when optical photons confined in a closed (lossless) cavity interact with the nonrelativistic movable totally reflective cavity boundaries. A Hamiltonian approach is usually applied to describe this kind of optomechanical interaction. In this paper, using an example of a one-dimensional (1D) Fabry-Perot cavity we show that the Hamiltonian approach is deficient if one considers an externally pumped cavity. The energy exchange between the cavity and the optical pumping strongly depends on the position of the mirror x, so the photon number in the optical mode changes significantly if the mirror motion is slow enough. This energy exchange dominates over high-order nonlinear by x phenomena observed in the case of closed (lossless) optical cavity, and this behavior cannot be predicted using a conventional optomechanical Hamiltonian. We show that the attenuation assisted nonlinearity can be so large that high-order mechanical harmonics can be readily generated in a mechanical system pumped with continuous wave light.

II. HAMILTONIAN APPROACH TO OPTOMECHANICS

Interaction of a single optical mode and a single mechanical degree of freedom can be presented in quasistatic approximation in form [15,24]

$$H_{\rm int} = -\hbar g \hat{a}^\dagger \hat{a} \hat{x}, \tag{1}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are photon creation and annihilation operators, $\hat{x}$ is a mechanical coordinate measured from the mechanical equilibrium point in the case of no light present, and g is an optomechanical coupling constant. In the case of a 1D Fabry-Perot cavity with a movable mirror (Fig. 1), this coupling constant is simply ω_0/L [15], where ω_0 is the carrier frequency of the light and L is the distance between the mirrors. The photon number does not change in this case. Motion of the mirror results in change of the optical frequency.

Hamiltonian (1) is not exact. To obtain it one has to utilize an adiabatic approximation in which the optical cavity is considered as a lumped system. The Hamiltonian also neglects by the nonlinear terms resulting from the change of the resonant optical frequency when the cavity dimension changes.

We are interested in nonlinear behavior of the optomechanical system and would like to derive a Hamiltonian that takes into account terms nonlinear in the mechanical coordinate $\hat{x}$.

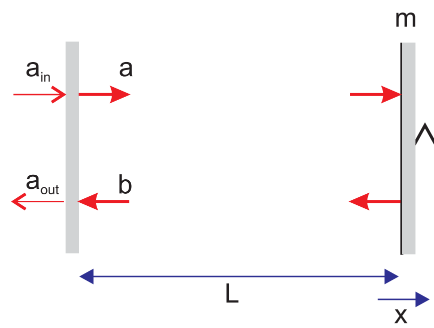

FIG. 1. Schematic of the 1D Fabry-Perot resonator with movable mirror.

The expression (1) directly follows from Maxwell equations. It is possible to write for electric field amplitude of light confined in an empty 1D Fabry-Perot cavity with totally reflecting mirrors (strictly speaking for steady state) the following equation:

$$\hat{a}(t) = \hat{a}[t - 2(L + \hat{x})/c], \tag{2}$$

where c is speed of light in the vacuum.

Assuming that $\hat{x}/L \ll 1$, in quasistatic approximation $L/c \gg \dot{\hat{x}}/x$, we can directly verify that expression

$$\hat{a} = \hat{a}(0)e^{\pm i\pi clt/(L+\hat{x})}, \tag{3}$$

(where l is the mode number) is a solution of Eq. (2). The quasistatic approximation is needed to prohibit photon exchange between the modes and to require photon conservation in the mode. Since the system is unitary, this is equivalent to saying that the mode frequency depends on coordinate as $\pi cl/(L + \hat{x})$ and the total Hamiltonian of a selected mode of the system is

$$H = \hbar\omega_0 \hat{a}^\dagger \hat{a} \frac{L}{L + \hat{x}}, \tag{4}$$

where the mode frequency is defined as $\omega_0 = \pi c l_0/L$ (l_0 is integer, corresponding wavelength $\lambda_0 = 2\pi c/\omega_0$). The interaction Hamiltonian, defined as $H_{\rm int} \equiv H - \hbar\omega_0 \hat{a}^\dagger \hat{a}$, becomes

$$H_{\rm int} = -\hbar g \hat{a}^\dagger \hat{a} \frac{\hat{x}}{1 + \hat{x}/L}. \tag{5}$$

In general, this Hamiltonian should be utilized instead of (1) to take the nonlinear terms $\hat{x}^n$ into account.

The Hamiltonian can be derived in a more explicit way using Eq. (2). Introducing slow amplitude $\hat{A}$, so that $\hat{a} = \hat{A}\exp(-i\omega_0 t)$, we rewrite Eq. (2) as

$$\hat{A}(t) = \hat{A}[t - 2(L + \hat{x})/c]e^{4i\pi\hat{x}/\lambda_0}. \tag{6}$$

The slow amplitude does not change much during the cavity round trip, which allows to use Taylor series

$$\hat{A}[t - 2(L + \hat{x})/c] \simeq \hat{A}(t) - \frac{2}{c}(L + \hat{x})\dot{\hat{A}} \tag{7}$$

to simplify Eq. (6),

$$\dot{\hat{A}} \simeq (1 - e^{-4i\pi\hat{x}/\lambda_0})\frac{c}{2(L + \hat{x})} \tag{8}$$

or, for the case of small mechanical amplitude $\lambda_0 \gg 4\pi|\langle|\hat{x}|\rangle|$ ($\langle \ldots \rangle$ stands for the expectation value), to a simpler differential equation

$$\dot{\hat{A}} \simeq i\omega_0 A \frac{\hat{x}}{\hat{x} + L}. \tag{9}$$

This equation is generated by Hamiltonian in the interaction picture

$$\tilde{H}_{\rm int} = -\hbar\omega_0 A^\dagger A \frac{\hat{x}}{L + \hat{x}}, \tag{10}$$

which is equivalent to Eq. (5) if $g = \omega_0/L$.

The Hamiltonian (5) results in the equation for the mechanical degree of freedom

$$\ddot{\hat{x}} + \omega_M^2\left[1 + \alpha_{om1}\left(1 - \frac{3}{2}\frac{\hat{x}}{L} + 2\frac{\hat{x}^2}{L^2}\right)\right]\hat{x} = \frac{\hbar g}{m}\hat{a}^\dagger\hat{a} + \frac{F_s(t)}{m}, \tag{11}$$

where we truncated the nonlinear terms of the order higher than $(\hat{x}/L)^3$ and introduced a classical mechanical force $F_s(t)$; m and ω_M are the mass and frequency of the mechanical system, respectively. To derive this equation, we first differentiate Eq. (5) by $\hat{x}$ and then decompose the result by powers of small parameter $\hat{x}/L$.

The nonlinearity of the system is defined by a dimensionless parameter

$$\alpha_{om1} = \frac{\hbar\omega_0 \hat{a}^\dagger \hat{a}}{m\omega_M^2 L^2}, \tag{12}$$

where we utilized $g = \omega_0/L$. The magnitude of α_{om1} is defined by the expectation value of the normalized dc shift of the mirror $\alpha_{om1} \sim 2\langle\hat{x}\rangle/L \ll 1$.

Nonlinear terms appearing in Eq. (5), $(\hat{x}/L)^n$, where $n > 1$ is an integer, can result in generation of higher-order mechanical harmonics if the size of the cavity is small enough. However, increase of the size to a kilometer range practically nullifies the effect. Moreover, the intrinsic mechanical nonlinearity of a micromechanical structure can be much larger if compared with the optomechanical part. For instance, similarly normalized mechanical nonlinearity parameter found from the Euler-Bernoulli theory applied to a microelectromechanical system (MEMS) cantilever can exceed unity by an order of magnitude [25–27]. The lossless cantilever motion obeys the equation (please see [25] for derivation)

$$\ddot{x} + \omega_M^2\left[1 + \frac{\beta_{\rm geom}}{m\omega_M^2}\frac{x^2}{\mathcal{L}^2} + \frac{\beta_{\rm iner}}{m\omega_M^2}\frac{\dot{x}^2 + x\ddot{x}}{\mathcal{L}^2}\right]x = \frac{F_s(t)}{m}, \tag{13}$$

where $\mathcal{L}$ is the cantilever length scaling in the micrometer range, $\beta_{\rm geom}$ and $\beta_{\rm iner}$ are geometrical and inertial nonlinear coefficients, respectively. It was shown that the effective dimensionless nonlinearity parameter $\alpha = [\beta_{\rm geom}/(m\omega_M^2) - 2\beta_{\rm iner}/(3m)]$ can exceed -20 for a real physical system. This is a much larger value if compared with the expected optomechanical nonlinearity α_{om1} involving reasonably small optical power. Therefore, it is reasonable to neglect by the ponderomotive mechanical nonlinearity in a unitary optomechanical system and consider only mechanical one.

III. OPEN OPTOMECHANICAL SYSTEM

We found, though, that there is a dissipation-associated mechanism that results in several orders of magnitude increase of the light-mitigated mechanical nonlinearity. The effect has common features with additional rigidity arising in an optomechanical system when a mechanical degree of freedom modulates the damping rate of a driven optical cavity [28,29]. In what follows, we derive the nonlinear terms using wave equation.

Let us consider an empty 1D Fabry-Perot resonator pumped with a plane wave $a_{in}(X,t) = A_{in}(X,t)\exp[-i(\omega t - kX)]$, where $k = \omega/c$ is the wave vector and X is the coordinate (Fig. 1). The front mirror of the resonator, characterized with the power transmission T, is placed at position $X_1 = 0$, and the back, total, mirror is movable, so its coordinate becomes $X_2 = L + \hat{x}(t)$, where L is the distance between the mirrors and $\hat{x}(t)$ is the time-dependent part of the total mirror coordinate. Standard equations describing electric field inside and outside of the resonator at the boundary of the input mirror ($X = 0$) are

$$\hat{a}(t) = \sqrt{1-T}\hat{b}(t) + i\sqrt{T}\hat{a}_{in}(t), \tag{14}$$

$$\hat{a}_{\rm out}(t) = i\sqrt{T}b(t) + \sqrt{1-T}\hat{a}_{in}(t), \tag{15}$$

$$\hat{b}(t) \simeq \hat{a}\left\{t - \frac{2[L + \hat{x}(t - L/c)]}{c}\right\}\left[1 - 2\frac{\dot{\hat{x}}(t)}{c}\right]. \tag{16}$$

Here, the term proportional to $\dot{\hat{x}}$ results from Doppler effect. While this term is usually small, it is necessary to keep it to sustain the right commutation relation for the coordinate and momentum of the mechanical system [30].

Substituting Eq. (16) to Eq. (14), we arrive to the equation for the field inside the resonator

$$\hat{a} - \sqrt{1-T}\hat{a}\left\{t - \frac{2[L + \hat{x}(t - L/c)]}{c}\right\}\left[1 - 2\frac{\dot{\hat{x}}(t)}{c}\right] = i\sqrt{T}\hat{a}_{in}. \tag{17}$$

Equation (17) coincides with Eq. (2) for the nonrelativistic case and closed (lossless, $T \equiv 0$) resonator.

Equation (17) has to be supplied with with equation for the coordinate of the movable mirror, that reads as

$$\ddot{\hat{x}}(t) + 2\gamma_M\dot{\hat{x}} + \omega_M^2\hat{x}(t) = \frac{\hbar\omega_0}{2mL}\left\{\hat{a}^\dagger\left[t - \frac{L + \hat{x}(t)}{c}\right]\hat{a}\left[t - \frac{L + \hat{x}(t)}{c}\right] + \hat{b}^\dagger\left[t + \frac{L + \hat{x}(t)}{c}\right]\hat{b}\left[t + \frac{L + \hat{x}(t)}{c}\right]\right\} + \frac{F_s(t)}{m}. \tag{18}$$

Here, force $F_s(t)$ includes both the signal and Langevin terms; mechanical attenuation γ_M is small.

In this equation, we notice that the ponderomotive force acting at the mirror results from the falling and reflecting light. In the particular case of the closed cavity, the photon number in the cavity does not change. In an open cavity, a part of the wave falling at the mirror can pass through the mirror, so we have to distinguish between $\hat{a}(t)$ and $\hat{b}(t)$. In addition, the time in our model is counted with respect of the light entering the system. It means that the photons that hit the movable mirror are delayed by the half of the round-trip time while the photons that reflect from the mirror are advanced by the half of the round-trip time. In other words, if at the fields, falling on and reflecting from front mirror, are described by operators $\hat{a}(t)$ and $\hat{b}(t)$, then on the back mirrors the fields are $\hat{a}(t - \tilde{L}/c)$ and $\hat{b}(t + \tilde{L}/c)$. The distance between the mirrors ($\tilde{L}$) also depends on time as the position of the back mirror changes while the light front propagates from one mirror to the other. That is why we consider the sum of the photon number at two different times (one retarded and one advanced) in the equation for the mechanical degree of freedom.

There are two general cases when set (17) and (18) can be simplified: $\omega_M L/c \ll 1$ and $\omega_M L/c = \pi j$, where j is a natural number. In the first case, the optomechanical interaction results in generation of optical harmonics localized within a single optical mode. In the second case, the mechanical frequency corresponds to the free spectral range of the resonator, so several optical modes (optical frequency comb) are generated due to the optomechanical interaction. For the case of a small optical cavity (a microcavity), the frequency of the mechanical mode is usually much smaller than the free spectral range of the cavity, so condition $\omega_M L/c \ll 1$ works. For the case of a large optical cavity, it is possible to find a configuration when $\omega_M L/c = \pi j$. In this paper, we consider both the cases. There are other configurations when the set of equations can be simplified. For instance, in a LIGO-type interferometer it is possible to find a mechanical mode that has frequency equal to the frequency difference of two optical modes belonging to two different mode families. We do not consider them here.

Let us introduce slow amplitude for the intracavity field $\hat{a}(X,t) = \hat{A}(X,t)\exp[-i(\omega t - kX)]$. The Taylor decomposition results in transformation of the equations involving operators depending on retarded time to standard ordinary differential equations [see Eq. (7) for the transformation details]. In the case of short enough optical cavity ($\hat{A} \gg L\dot{\hat{A}}/c$) we derive from Eq. (17) a simplified equation for the slow intracavity field amplitude

$$\dot{\hat{A}} + [\Gamma(\hat{x}) - i\Delta(\hat{x})]\hat{A} = \frac{i\sqrt{T}}{\tau}\hat{A}_{in}, \tag{19}$$

where the coordinate-dependent optical attenuation and dispersion are given by formulas

$$\Gamma(\hat{x}) = \frac{1}{\tau}(1 - \sqrt{1-T}\cos\{2k[L + \hat{x}(t)]\}), \tag{20}$$

$$\Delta(\hat{x}) = \frac{1}{\tau}\sqrt{1-T}\sin\{2k[L + \hat{x}(t)]\}, \tag{21}$$

$\tau = 2L/c$ is the cavity round-trip time. Neglecting by the small terms associated with the Doppler effect as well as assuming $\omega_M\tau \ll 1$ we also simplify Eq. (18) for the mechanical system

$$\ddot{\hat{x}} + 2\gamma_M\dot{\hat{x}} + \omega_M^2\hat{x} = \frac{\hbar\omega_0}{mL}\hat{A}^\dagger(t)\hat{A}(t) + \frac{F_s(t)}{m}. \tag{22}$$

To solve this set of equations, we assume that $F_s(t)$ is small and look for the solution in the vicinity of steady state defined by expectation values for the field and mechanical amplitudes

$$A \simeq \frac{i\sqrt{T}}{\tau}\frac{\hat{A}_{in}}{\Gamma_0 - i\Delta_0}, \tag{23}$$

$$x_0 \simeq \frac{\hbar\omega_0}{m\omega_M^2 L}|A|^2, \tag{24}$$

where

$$\Gamma_0 = \frac{1}{\tau}\{1 - \sqrt{1-T}\cos[2k(L+x_0)]\}, \tag{25}$$

$$\Delta_0 = \frac{1}{\tau}\sqrt{1-T}\sin[2k(L+x_0)]. \tag{26}$$

It is also assumed for convenience that x_0 includes all the smaller order dc terms appearing during the analysis of the nonlinear system. In the following analysis, we consider only time-dependent part of coordinate $\delta\hat{x} = \hat{x} - x_0$.

General analysis of the optomechanical system is rather involved. We are interested in evaluation of the nonlinear response and consider the exact resonant case ($\Delta_0 = 0$). We formally solve Eq. (19) for the field amplitude and substitute the solution into Eq. (18). Linear in the coordinate terms responsible for the well-known ponderomotive attenuation and rigidity disappear for the resonant tuning of the pump light. The cubic nonlinearity terms also proportional to the optical detuning disappear as well. Only quadratic in coordinate terms survive.

The nonlinear equation for the mechanical degree of freedom with excluded optical variables can be presented in the form (see Appendix)

$$\ddot{\delta\hat{x}} + 2\gamma_M\dot{\delta\hat{x}} + \omega_M^2\left[1 + \alpha_{om2}\frac{\delta\hat{x}}{L}\right]\delta\hat{x} = \frac{F_s}{m}, \tag{27}$$

where the dimensionless quadratic nonlinearity parameter α_{om2} depends on the frequency of the forced oscillation. For instance, for the case of resonant mechanical force ($F_s = f_s\cos\omega_M t$) and relatively low quality factor of the optical cavity ($\Gamma_0 \gg \omega_M$) the nonlinearity parameter is

$$\alpha_{om2} = -4\frac{\hbar\omega_0|A|^2}{m\omega_M^2}\frac{Q^2}{L^2}. \tag{28}$$

It is obtained using expression $Q = \omega_0/(2\Gamma_0)$ for the optical quality factor.

Equation (11) for the mechanical coordinate obtained for the closed (unitary) optomechanical system also contains a quadratic term α_{om1} [Eq. (12)] which is $4Q^2 \gg 1$ times smaller than α_{om2}. Therefore, to find the nonlinearity in a correct way, the unitary model has to be adjusted.

An approximate solution of the equation with respect to the expectation value of coordinate is

$$\delta x \simeq \frac{f_s}{2m\gamma_M\omega_M}\sin(\omega_M t) + \frac{\alpha_{om2}}{L}\left(\frac{f_s}{2m\gamma_M\omega_M}\right)^2\cos(2\omega_M t), \tag{29}$$

where we omitted the zero-frequency term assuming it to be a part of x_0. Equation (29) shows that analysis of the mechanical spectrum allows evaluating the optomechanical nonlinearity.

For some practical applications it is useful to consider the case of high-frequency force $F_s = f_s\cos(\omega_f t)$, where $\omega_f \gg \omega_m$, but $\omega_f\tau \ll 1$. In this case the nonlinearity reduces, but still is large:

$$\delta x \simeq -\frac{f_s}{m\omega_f^2}\sin(\omega_f t) + \frac{\omega_M^2}{16\omega_f^2 L}\left[\alpha_{om2}^{\rm fm}e^{2i\omega_f t} + \alpha_{om2}^{\rm fm*}e^{-2i\omega_f t}\right]\left(\frac{f_s}{m\omega_f^2}\right)^2, \tag{30}$$

$$\alpha_{om2}^{\rm fm} = 4\frac{\hbar\omega_0|A|^2}{m\omega_M^2}\frac{Q^2}{L^2}S, \quad S = \frac{-\Gamma_0^3}{(\Gamma + i\omega_f)^2(\Gamma_0 + 2i\omega_f)}. \tag{31}$$

Presence of the strong quadratic optomechanical nonlinearity contrasts with the absence of the similar term in the purely mechanical nonlinearity of the system. The physical nature of this optomechanical nonlinearity is related to the reduction of the intracavity power when the system deviates from the optical resonance. The power drops independently on the direction of the mechanical motion.

The pure mechanical nonlinearity is of cubic nature [Eq. (13)]. The nonlinearity of the unitary system contain a small cubic part $2\alpha_{om1}$ for the normalization selected in Eq. (11). The cubic nonlinearity terms are also introduced to the open optomechanical system for $\Delta_0 \sim \Gamma_0$. Omitting lengthy derivations, we write for the corresponding cubic nonlinear coefficient

$$\alpha_{om3} \simeq \frac{k^3 L\hbar\omega_0|A|^2}{m\omega_M^2 T^3}. \tag{32}$$

It is easy to see that this nonlinearity is $k^3L^3/T^3 \gg 1$ times larger than the nonlinearity α_{om1} of the optically closed (lossless) optomechanical system. The reason for the nonlinearity enhancement is again the interaction of the optomechanical system with continuum resulting in the change of the optical power in the cavity when the position of the mirror changes.

The magnitude α_{om3} can exceed the unity and be comparable with MEMS nonlinearity parameter α for a small number of optical photons in the cavity. Really, for an optomechanical system with MEMS mirror we get $\alpha_{om3} \simeq 10^3$ for $\lambda = 532$ nm, $L = 0.1$ cm, $|A|^2 = 10^2$, $m = 1$ mg, $\omega_M = 2\pi\times 1$ MHz, and $T = 10^{-3}$.

The results of our calculations have qualitative match with experimental data. Optomechanical systems used to demonstrate generation of multiple equidistant optical harmonics separated by the mechanical frequency. The neighboring harmonics are approximately of the same magnitude. It means that the system has both strong odd and even nonlinear terms. Pure mechanical nonlinearity tends to have mostly odd terms. Presence of even terms is also possible if the system is prestressed, however, their magnitude is usually small. Presence of the significant quadratic nonlinearity of the optomechanical system explains observed experimentally efficient generation of the optical sidebands separated from the pump carrier by the doubled mechanical frequency.

IV. FREE-MASS INTERFEROMETER

It is interesting to estimate the optomechanical nonlinearity in the case of $\omega_M \to 0$ since the nonlinearity increases with

ω_M decrease. Such a configuration is practically realized in the Advanced Laser Interferometric Gravitational Observatory (aLIGO) [31,32] which can be reduced to an equivalent 1D Fabry-Perot cavity [33] (corresponding to so-called signal recycling mode) with movable mirror having mass $m = 10$ kg and frequency $\omega_M/2\pi \sim 0.1$ Hz. The bandwidth of the optical cavity is about working bandwidth of aLIGO (it is varied by position of signal recycling mirror), in estimates below we assume $\Gamma_0/2\pi \sim 300$ Hz. The zero- and first-order optomechanical effects are very important in this case. The zero-order ponderomotive effect associated with the radiation pressure results in accelerated motion of the mirror that cannot be tolerated. To handle this effect, an electronic feedback is involved [31,34]. Because of the feedback loops, the optomechanical system cannot be considered using the simplest model presented above, however, the mirror can be treated as a free mass in 30–1000 Hz frequency range.

We can use Eq. (31) to evaluate nonlinearity in this case for LIGO parameters [31]. Selecting $\Gamma_0 = 2\pi\times 300$ rad/s, $\omega_f = 2\pi\times 10^2$ rad/s, $P = 800$ kW, $m = 10$ kg (reduced mass), $L = 4$ km, $\hbar\omega_0|A|^2 = 2LP/c$, $\lambda = 1064$ nm, we arrive at

$$\frac{|\alpha_{om2}^{\rm fm}|\omega_M^2}{8\omega_f^2 L} = Q^2\left(\frac{P}{m\omega_f^2 L^2 c}\right)|S| \simeq 8\times 10^6 \ \ {\rm m}^{-1}. \quad (33)$$

In other words, if the magnitude of the first mechanical harmonic is $f_s/(m\omega_f^2) = 0.01$ nm, the magnitude of the second mechanical harmonic is about 8×10^{-16} m.

This can be easily detected in Advanced LIGO [31,32,35]. The unitary model predicts the magnitude to be many orders of magnitude smaller, which is practically undetectable in the system.

V. CONCLUSION

In this paper, we have shown theoretically that optomechanical nonlinearity induced due to the open nature of the system can be much larger if compared with the nonlinearity of an optically closed (lossless) optomechanical system having the same other parameters. The effect arises due to the variation of the intracavity photon number in the open system as a function of the mechanical coordinate. In contrast, the photon number of the lossless optomechanical system is conserved and only the frequency of the photons changes due to variations of the mechanical degree of freedom. We found that the mechanical nonlinearity induced by the optical degree of freedom can be comparable with purely mechanical nonlinearity both in small scale for micromechanical cantilevers and in large scale for 40-kg free masses (mirrors) in Advanced LIGO interferometer.

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APPENDIX

We provide derivation of Eqs. (19) and (27) in this appendix. Let us start from Eq. (17) and introduce slow field amplitudes

$$\hat a = \hat A \exp[-i(\omega t - kX)]. \quad (A1)$$

Assuming nearly resonant tuning of the optical system $[\exp(ikL) \simeq 1]$, neglecting by the Doppler term $(\sim\dot{\hat x}/c)$, assuming that the mechanical frequency is small enough, $T/2\tau \gg \omega_M$ [so that $\hat x(t-\tau) \sim \hat x(t)$], and using Taylor series

$$\hat A(t-\tau) \simeq \hat A(t) - \tau\dot{\hat A}(t) \quad (A2)$$

we arrive at

$$\dot{\hat A}\underbrace{\sqrt{1-T}[1-\mathbb{D}(t-\tau)]}_{\simeq 1} + \frac{\hat A}{\tau}\{1-\sqrt{1-T}[1-\mathbb{D}(t-\tau)]\}$$
$$= \frac{i\sqrt{T}\,\hat A_{in}(t)}{\tau}, \quad \mathbb{D} = 1 - e^{2ik\hat x(t-\tau)} \simeq 1 - e^{2ik\hat x(t)}. \quad (A3)$$

Equation (A3) results in Eqs. (19) and (21).

To find the nonlinearity introduced to the mechanical degree of freedom by the optical degree of freedom, we decompose the mechanical coordinate into time-independent and time-dependent parts

$$\hat x(t) = x_0 + \delta\hat x(t). \quad (A4)$$

In this case,

$$\Gamma(\hat x) \approx \Gamma_0 + \omega_0\frac{k(\delta\hat x)^2}{L},$$
$$\Delta(\hat x) \approx \Delta_0 + \omega_0\frac{\delta\hat x}{L}. \quad (A5)$$

For the sake of simplicity, we consider a particular case and assume that the mechanical force is monochromatic $[F_s = f_s\cos(\omega_M t)]$, that optical pump is resonant $\Delta_0 = 0$, and that the optical pump is classical and its phase selected so that $i\hat A_{in} = |A_{in}|$. The optical and mechanical amplitudes can be presented in a form of decomposition by the harmonics of the frequency defined by the mechanical force

$$A = \sum_{j=-\infty}^{\infty} A_j e^{ij\omega_M t},$$
$$\delta x = \sum_{j=-\infty}^{\infty} x_j e^{ij\omega_M t}. \quad (A6)$$

Substituting the decompositions into the nonlinear equations, we obtain

$$A_{+1} \simeq i\omega_0\frac{x_{+1}}{L(\Gamma_0 + i\omega_M)}A_0, \quad (A7)$$

$$A_{-1} \simeq i\omega_0\frac{x_{-1}}{L(\Gamma_0 - i\omega_M)}A_0. \quad (A8)$$

Because we assumed that $A_0 = A_0^*$ and since $x_{-1}^* = x_{+1}$, we have $A_{-1}^* = -A_{+1}$.

For the second-order harmonics we derive

$$A_{+2} \simeq i\omega_0 \frac{x_{+2}A_0}{L(\Gamma_0 + 2i\omega_M)} - \frac{\omega_0^2 x_{+1}^2}{L^2(\Gamma_0 + i\omega_M)(\Gamma_0 + 2i\omega_M)} A_0, \quad \text{(A9)}$$

$$A_{-2} \simeq i\omega_0 \frac{x_{-2}A_0}{L(\Gamma_0 - 2i\omega_M)} - \frac{\omega_0^2 x_{-1}^2}{L^2(\Gamma_0 - i\omega_M)(\Gamma_0 - 2i\omega_M)} A_0. \quad \text{(A10)}$$

Using similar reasoning, we obtain expressions for mechanical harmonics

$$x_{+1} \simeq \frac{\hbar\omega_0}{2i\gamma_M\omega_M m L} A_0(A_{+1} + A_{-1}^*) + f_+, \quad \text{(A11)}$$

$$x_{-1} \simeq -\frac{\hbar\omega_0}{2i\gamma_M\omega_M m L} A_0(A_{-1} + A_{+1}^*) + f_-, \quad \text{(A12)}$$

$$f_\pm \equiv \frac{\pm f_s}{4i\gamma_M\omega_M m}, \quad \text{(A13)}$$

$$x_{+2} \simeq -\frac{\hbar\omega_0}{3mL\omega_M^2}(A_{-1}^* A_{+1} + A_0 A_{+2} + A_0 A_{-2}^*) = -\frac{\hbar\omega_0}{3mL\omega_M^2}(A_{-1}^* A_{+1} + A_0[A_{+2} + A_{-2}^*]), \quad \text{(A14)}$$

$$x_{-2} \simeq -\frac{\hbar\omega_0}{3mL\omega_M^2}(A_{-1} A_{+1}^* + A_0 A_{+2}^* + A_0 A_{-2}) = -\frac{\hbar\omega_0}{3mL\omega_M^2}(A_{-1} A_{+1}^* + A_0[A_{+2}^* + A_{-2}]). \quad \text{(A15)}$$

Using the expressions presented above, we derive expressions for the first and second harmonics of the mechanical and optical amplitudes

$$A_{+1} = -A_{-1}^* = \frac{f_s}{4\gamma_M\omega_M m L}\frac{\omega_0}{\Gamma_0 + i\omega_M}, \quad \text{(A16)}$$

$$x_{+1} = x_{-1}^* = \frac{f_s}{4i\gamma_M\omega_M m}; \quad \text{(A17)}$$

and

$$x_{+2} = x_{-2}^* = -\frac{\hbar\omega_0}{3mL\delta^2}(-\beta^2 + 2\beta\beta_2) f_+^2, \quad \text{(A18)}$$

$$A_{+2} = \beta_2 x_{+2} + \frac{\beta\beta_2 f_+^2}{A_0}, \quad \text{(A19)}$$

$$A_{-2} = -\beta_2^* x_{-2} + \frac{\beta^*\beta_2^* f_-^2}{A_0}, \quad \text{(A20)}$$

$$\beta = \frac{i\omega_0 A_0}{L(\Gamma_0 + i\omega_M)}, \quad \text{(A21)}$$

$$\beta_2 = \frac{i\omega_0 A_0}{L(\Gamma_0 + 2i\omega_M)}. \quad \text{(A22)}$$

Comparing these expressions with Eq. (29), we get Eq. (28) for α_{om2}.

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