Combinatorial structure of root systems for Lie (super)algebras and its applications

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The International School
"Advanced Methods of Modern Theoretical Physics:
Integrable and Stochastic Systems"
(Dubna, Russia, August 16-21, 2015)

Contents

Table 1. (Super)algebras of rank 1.

$g(A,\tau)$	A	odd roots	diagram	dim	Δ_+	$\underline{\Delta}_+$
A_1	(2)	$\emptyset$	$\bigcirc$	3	α	α
$B(0,1)$	(2)	$\{\alpha\}$	$\bullet$	5	$\alpha, 2\alpha$	α
$sl(1,1)$	(0)	$\{\alpha\}$	$\otimes$	3	α	α
$g((0),\{1\})$	(0)	$\{\alpha\}$	$\otimes$	4	α	α
$g((0),\emptyset)$	(0)	$\emptyset$	$\odot$	4	α	α

Table 2a. (Super)algebras of rank 2.

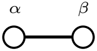
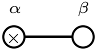
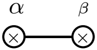
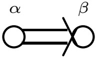
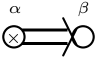
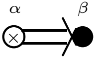
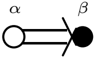
$g(A,\tau)$	A	A^{sym}	$(A^{sym})^{-1}$	odd	diagram	dim
A_2	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	$\emptyset$		8
$A(1,0)$	$\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} -0 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix}$	$\{\alpha\}$		8
$A'(1,0)$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\{\alpha,\beta\}$		8
B_2	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	$\emptyset$		10
$B(1,1)$	$\begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$	$\{\alpha\}$		12
$B'(1,1)$	$\begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$	$\{\alpha,\beta\}$		12
$B(0,2)$	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	$\{\beta\}$		14

Table 2b. (Super)algebras of rank 2.

$g(A,\tau)$	Δ_+	$\underline{\vec{\Delta}}_+$
A_2	$\alpha, \alpha+\beta, \beta$	$\alpha, \alpha+\beta, \beta$ $\beta, \alpha+\beta, \alpha$
$A(1,0)$	$\alpha, \alpha+\beta, \beta$	$\alpha, \alpha+\beta, \beta$ $\beta, \alpha+\beta, \alpha$
$A'(1,0)$	$\alpha, \alpha+\beta, \beta$	$\alpha, \alpha+\beta, \beta$ $\beta, \alpha+\beta, \alpha$
B_2	$\alpha, \alpha+\beta, \alpha+2\beta, \beta$	$\alpha, \alpha+\beta, \alpha+2\beta, \beta$ $\beta, \alpha+2\beta, \alpha+\beta, \alpha$
$B(1,1)$	$\alpha, \alpha+\beta, 2\alpha+2\beta, \alpha+2\beta, \beta$	$\alpha, \alpha+\beta, \alpha+2\beta, \beta$ $\beta, \alpha+2\beta, \alpha+\beta, \alpha$
$B'(1,1)$	$\alpha, \alpha+\beta, \alpha+2\beta, 2\beta, \beta$	$\alpha, \alpha+\beta, \alpha+2\beta, \beta$ $\beta, \alpha+2\beta, \alpha+\beta, \alpha$
$B(0,2)$	$\alpha, \alpha+\beta, 2\alpha+2\beta, \alpha+2\beta, 2\beta, \beta$	$\alpha, \alpha+\beta, \alpha+2\beta, \beta$ $\beta, \alpha+2\beta, \alpha+\beta, \alpha$
G_2	$\alpha, \alpha+\beta, 2\alpha+3\beta, \alpha+2\beta, \alpha+3\beta, \beta$	$\alpha, \alpha+\beta, 2\alpha+3\beta, \alpha+2\beta, \alpha+3\beta, \beta$ $\beta, \alpha+3\beta, \alpha+2\beta, 2\alpha+3\beta, \alpha+\beta, \alpha$

Table 3a. Affine (super)algebras of rank 2.

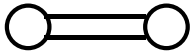
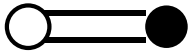
$g(A,\tau)$	A	A^{sym}	$\bar{A}^{sym}$	odd	diagram
$A_1^{(1)}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$	$\emptyset$	$\delta-\alpha$ α 
$C(2)^{(2)}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} -0 & -1 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$	$\{\delta-\alpha,\alpha\}$	$\delta-\alpha$ α 
$A(0,2)^{(4)}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$	$\{\alpha\}$	$\delta-\alpha$ α 
$A_2^{(2)}$	$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$	$\begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 8 & -4 \\ 0 & -4 & -2 \end{pmatrix}$	$\emptyset$	$\delta-2\alpha$ α 
$B(0,1)^{(1)}$	$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$	$\begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 8 & -4 \\ 0 & -4 & 2 \end{pmatrix}$	$\{\alpha\}$	$\delta-2\alpha$ α 

Table 3a. Affine (super)algebras of rank 2.

$g(A,\tau)$	Δ_+	$\underline{\Delta}_+$
$A_1^{(1)}$	$\{\alpha,\ n\delta\pm\alpha,\ n\delta \mid n\in\mathbb{N}\}$	$\{\alpha,\ n\delta\pm\alpha,\ n\delta \mid n\in\mathbb{N}\}$
$C(2)^{(2)}$	$\{\alpha,\ 2\alpha,\ n\delta\pm\alpha,\ 2n\delta\pm 2\alpha,\ n\delta \mid n\in\mathbb{N}\}$	$\{\alpha,\ n\delta\pm\alpha,\ n\delta \mid n\in\mathbb{N}\}$
$A(0,2)^{(4)}$	$\{\alpha,\ 2\alpha,\ n\delta\pm\alpha,\ 2n\delta\pm 2\alpha,\ n\delta \mid n\in\mathbb{N}\}$	$\{\alpha,\ n\delta\pm\alpha,\ n\delta \mid n\in\mathbb{N}\}$
$A_2^{(2)}$	$\{\alpha,\ n\delta\pm\alpha,\ (2n-1)\delta\pm 2\alpha,\ n\delta \mid n\in\mathbb{N}\}$	$\{\alpha,\ n\delta\pm\alpha,\ (2n-1)\delta\pm 2\alpha,\ n\delta \mid n\in\mathbb{N}\}$
$B(0,1)^{(1)}$	$\{\alpha,\ n\delta\pm\alpha,\ n\delta\pm 2\alpha,\ n\delta \mid n\in\mathbb{N}\}$	$\{\alpha,\ n\delta\pm\alpha,\ (2n-1)\delta\pm 2\alpha,\ n\delta \mid n\in\mathbb{N}\}$

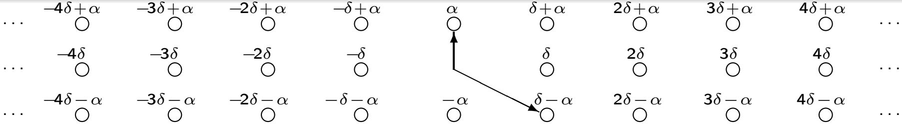


Fig 1. Total and reduced root system ($\Delta_+ = \underline{\Delta}_+$) of the superalgebra $A_1^{(1)} (\simeq \widehat{sl}_2)$

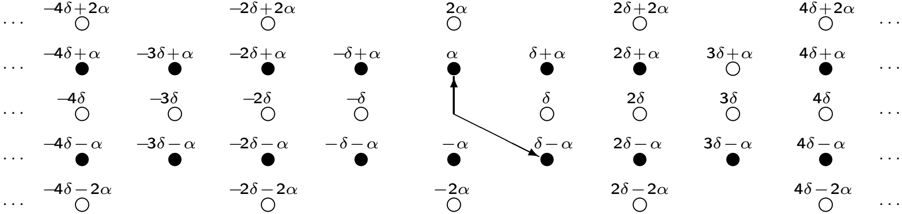
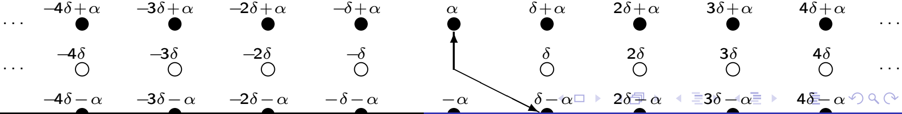


Fig 2a. The total root system Δ_+ of the superalgebra $C(2)^{(2)}$



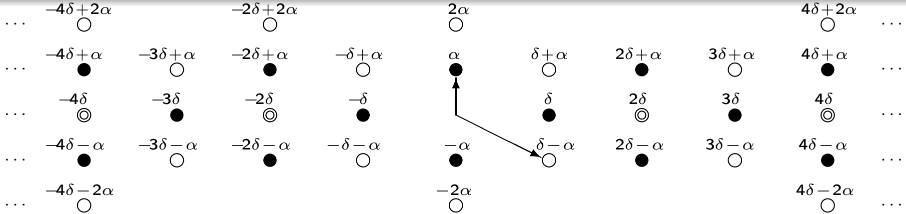


Fig 3a. The total root system Δ_+ of the superalgebra $A(0,2)^{(\mathbf{4})}$

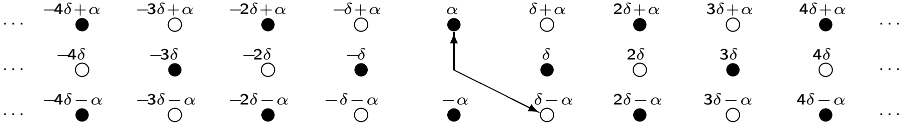


Fig 3b. The reduced root system $\underline{\Delta}_+$ of the superalgebra $A(0,2)^{(\mathbf{4})}$

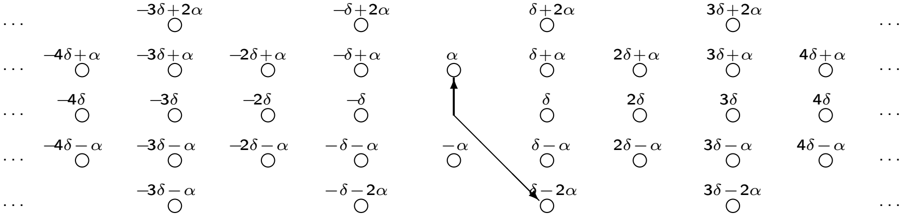


Fig 4. The total and reduced root system ($\Delta_+ = \underline{\Delta}_+$) of the superalgebra $A_2^{(2)}$

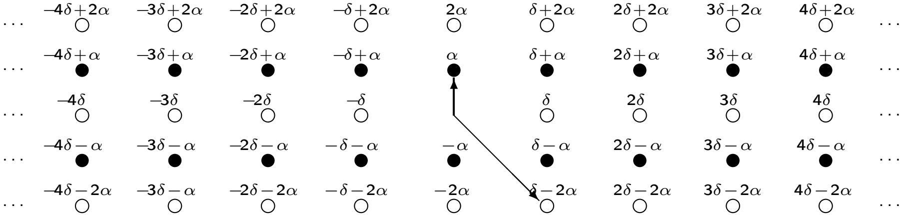


Fig 5a. The total root system Δ_+ of the superalgebra $B(0,1)^{(1)}$

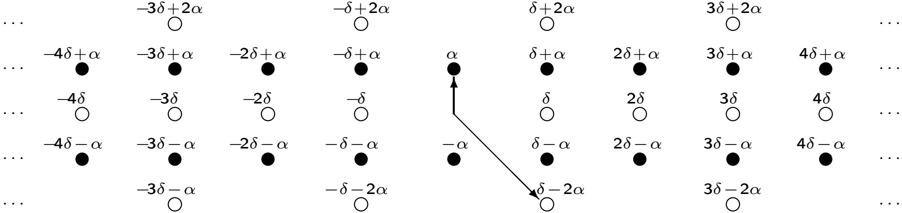


Fig 5b. The reduced root system $\underline{\Delta}_+$ of the superalgebra $B(0,1)^{(1)}$

$$\alpha,\, \delta+\alpha,\, 2\delta+\alpha,\, \ldots,\, \infty\delta+\alpha,\, \delta,\, 2\delta,\, 3\delta,\, \ldots,\, \infty\delta,\, \infty\delta-\alpha,\, \ldots,\, 3\delta-\alpha,\, 2\delta-\alpha,\, \delta-\alpha, \tag{1}$$

$$\delta-\alpha,\, 2\delta-\alpha,\, 3\delta-\alpha,\, \ldots,\, \infty\delta-\alpha,\, \delta,\, 2\delta,\, 3\delta,\, \ldots,\, \infty\delta,\, \infty\delta+\alpha,\, \ldots,\, 2\delta+\alpha,\, \delta+\alpha,\, \alpha. \tag{2}$$

$$\begin{gathered}\alpha,\, \delta+2\alpha,\, \delta+\alpha,\, 3\delta+2\alpha,\, 2\delta+\alpha,\, \ldots,\, \infty\delta+\alpha,\, (2\infty+1)\delta+2\alpha,\, (\infty+1)\delta+\alpha,\, \delta,\, 2\delta,\, \ldots, \\ \infty\delta,\, (\infty+1)\delta-\alpha,\, (2\infty+1)\delta-2\alpha,\, \infty\delta-\alpha,\, \ldots,\, 2\delta-\alpha,\, 3\delta-2\alpha,\, \delta-\alpha,\, \delta-2\alpha,\end{gathered} \tag{3}$$

$$\begin{gathered}\delta-2\alpha,\, \delta-\alpha,\, 3\delta-2\alpha,\, 2\delta-\alpha, \ldots, \infty\delta-\alpha,\, (2\infty+1)\delta-2\alpha,\, (\infty+1)\delta-\alpha,\, \delta,\, 2\delta,\, \ldots, \\ \infty\delta,\, (\infty+1)\delta+\alpha,\, (2\infty+1)\delta+2\alpha,\, \infty\delta+\alpha,\, \ldots,\, 2\delta+\alpha,\, 3\delta+2\alpha,\, \delta+\alpha,\, \delta+2\alpha,\, \alpha.\end{gathered} \tag{4}$$

Let G be a finite and compact group, and T be its representation in a linear space V, i.e. $g \mapsto T(g)$, $(g \in G)$, where $T(g)$ is a linear operator acting in V, provided that $T(g_1 g_2) = T(g_1)T(g_2)$. The representation T in V is irreducible if $Lin\{T(G)v\} = V$ for any nonzero vector $v \in V$. A irreducible representation (IR) is denoted by the additional upper index λ, $T^{\lambda}(g)$, and also V^{λ}, or in a matrix form: $(T^{\lambda}(g)) = (t_{ij}^{\lambda}(g))$ $(i, j = 1, 2, \ldots, n)$, where n is the dimension of V^{λ}. It is well-known that the elements

$$P_{ij}^{\lambda} = \sum_{g \in G} T(g) t_{ij}^{\lambda}(g) \tag{5}$$

are projection operators for the finite group G, i.e. they satisfy the following properties:

$$P_{ij}^{\lambda} P_{kl}^{\lambda'} \;=\; \delta_{\lambda\lambda'} \delta_{jk} P_{il}^{\lambda}, \qquad (P_{ij}^{\lambda})^{*} = P_{ji}^{\lambda}, \tag{6}$$

where * is the Hermitian conjugation.

The projection operators for a compact group G are modified as follows:

$$P_{ij}^{\lambda} = \int\limits_{g\in G} T(g)t_{ij}^{\lambda}(g)dg \ . \tag{7}$$

In the case $G = SO(3)$ (or $SU(2)$) we have

$$P^{j}_{mm'} = \int T(\alpha,\beta,\gamma)\, D^{j}_{mm'}(\alpha,\beta,\gamma) \sin\beta\, d\alpha\, d\beta\, d\gamma \ , \tag{8}$$

where α, β, γ are the Euler angles and $D^{j}_{mm'}(\alpha,\beta,\gamma)$ is the Wigner D-function. The projection operator $P^{j} := P^{j}_{jj}$ is called the projector on the highest weight j.
Thus we see that the projection operators in the form (**??**) or (**??**) demands the explicit expressions for the operator function $T(g)$, the matrix elements of IRs $t_{ij}^{\lambda}(g)$, and also (in the case of a compact group) the g-invariant measure dg. In the case of arbitrary compact group G these expressions lead to several problems.

The angular momenta Lie algebra $\mathfrak{so}(3)$ ($\simeq \mathfrak{su}(2)$) is generated by the three elements (generators) J_+, J_- and J_0 with the defining relations:

$$\begin{aligned} [J_0, J_\pm] &= \pm J_\pm, \qquad [J_+, J_-] = 2J_0, \\ J_\pm^* &= J_\mp\ , \qquad\qquad J_0^* = J_0\ . \end{aligned} \tag{9}$$

The Casimir element $\mathbf{C}_2$ of the angular momenta Lie algebra (or square of the angular momenta $\mathbf{J}^2$) is given by:

$$\mathbf{C}_2 \equiv \mathbf{J}^2 = \frac{1}{2}\Big(J_+J_- + J_-J_+\Big) + J_0^2 = J_-J_+ + J_0(J_0+1)\ , \tag{10}$$

$$[J_i\ , \mathbf{J}^2] = 0\ . \tag{11}$$

Let $\{|jm\rangle\}$ be the canonical basis of $\mathfrak{su}(2)$-IR corresponding to the spin j (wave functions with the definite j and its projections m ($m=-j,-j+1,\ldots,j$), for example, spherical harmonics, $|jm\rangle\equiv Y_m^j$). These basis functions satisfy the relations:

$$\begin{aligned} \mathbf{J}^2|jm\rangle &= j(j+1)|jm\rangle, \quad J_0|jm\rangle \;=\; m|jm\rangle \;, \\ J_\pm|jm\rangle &= \sqrt{(j\mp m)(j\pm m+1)}\;|jm\pm 1\rangle \;. \end{aligned} \tag{12}$$

The vectors $|jm\rangle$ can be represented as follows

$$|jm\rangle = F^{\,j}_{m;j}|jj\rangle \;, \tag{13}$$

where

$$F^{\,j}_{m;j} = \sqrt{\frac{(j+m)!}{(2j)!(j-m)!}}\;J_-^{j-m}, \quad \Bigl(F^{\,j}_{j;m} := (F^{\,j}_{m;j})^*\Bigr), \tag{14}$$

and $|jj\rangle$ is the highest weight vector, i.e.

$$J_+|jj\rangle = 0 \;. \tag{15}$$

It is obvious that the projector P^j on the highest weight,
$P^j = \int T(\alpha,\beta,\gamma)\, D^j_{jj}(\alpha,\beta,\gamma) \sin\beta d\alpha d\beta d\gamma$, satisfies the relations:

$$J_+ P^j = P^j J_- = 0, \qquad (P^j)^2 = P^j \, . \tag{16}$$

An associative polynomial algebra of the generators $J_{\pm}$, J_0 is called the universal enveloping algebra of the angular momenta Lie algebra and it is denoted by $U(\mathfrak{so}(3))$ (or $U(\mathfrak{su}(2))$. The following proposition holds.
"No-go theorem": *No nontrivial solution of the equations*

$$J_+ P = P J_- = 0 \tag{17}$$

exists in $U(\mathfrak{su}(2))$, i.e. a unique solution of these equations for $P \in U(\mathfrak{su}(2))$ is trivial $P \equiv 0$.
Thus the theorem states that the projector P^j does not exist in the form of a polynomial of the generators $J_{\pm}$, J_0. This no-go theorem was well known to mathematicians but we can assume that it was not known to most of the physicists.

In 1964 Swedish physicist and chemist P.-O. Lӧwdin, who probably did not know the no-go theorem, published the paper in Rev. Mod. Phys.(v.36, 966), where he considered the following operator:

$$P^j := \prod_{j'\neq j} \frac{\mathbf{J}^2 - j'(j'+1)}{j(j+1) - j'(j'+1)} \ . \tag{18}$$

This element has the following properties. Let $\Psi_{m=j}$ be arbitrary eigenvector of the operator J_0:

$$J_0\Psi_m = m\Psi_m \ . \tag{19}$$

Due to completeness of the basis formed by the vectors $\{|jm\rangle\}$ (for all possible spins j and their projections m) the expansion holds:

$$\Psi_m = \sum_{j'} C_{j'}|j'm\rangle \ , \tag{20}$$

and it is obvious that

$$P^j\Psi_{m=j} = C_j|jj\rangle \ . \tag{21}$$

From here we obtain the following properties of the element P^j:

$$J_+P^j = P^j J_- = 0 , \tag{22}$$

$$[J_0, P^j] = 0 , \qquad (P^j)^2 = P^j , \tag{23}$$

provided that the left and right sides of these equalities act on vectors with the definite projection of angular momentum $m = j$. Therefore the element P^j is the projector on the highest weight.

After rather complicated calculations Löwdin reduced the operator $P^j := \prod_{j'\neq j} \frac{\mathbf{J}^2 - j'(j'+1)}{j(j+1)-j'(j'+1)}$ to the following form:

$$P^j = \sum_{n\geq 0} \frac{(-1)^n(2j+1)!}{n!(2j+n+1)!} J_-^n J_+^n . \tag{24}$$

One year later, in 1965, another physicist J. Shapiro from USA published the paper in J. Math. Phys. (v.6, 1680), where he said: "Let us forget the initial expression (**??**) and consider the defining relations (**??**) and (**??**), where P^j has the following ansatz:

$$P^j = \sum_{n\geq 0} C_n(j) J_-^n J_+^n \ ." \tag{25}$$

Substituting this expression in the defining relation $J_+ P^j = 0$ we directly obtain the previous formula (**??**).
We can remove the upper index j in P^j if we replace $j \to J_0$:

$$P = \sum_{n\geq 0} \frac{(-1)^n}{n!} \varphi_n(J_0) J_-^n J_+^n \ , \quad \varphi_n(J_0) = \prod_{k=1}^{n} (2J_0 + k + 1)^{-1} \ . \tag{26}$$

The element P is called the extremal projector. If Ψ is an arbitrary function $\Psi = \sum_{j,m} C_{j,m}|jm\rangle$ then

$$P\Psi = \sum_j C_{j,j}|jj\rangle \ . \tag{27}$$

The extremal projector P does not belong to $U(\mathfrak{su}(2))$ but it belongs to some extension of the universal enveloping algebra.
Let us determine this extension. Consider the formal Taylor series

$$\sum_{n,k\geq 0} C_{n,k}(J_0)\, J_-^n\, J_+^k \tag{28}$$

provided that

$$|n-k|<N, \quad \text{for some } N, \tag{29}$$

where $C_{n,k}(J_0)$ are rational functions of the Cartan element J_0.
Let $TU(\mathfrak{su}(2))$ be a linear space of such formal series.
We can show that $TU(\mathfrak{su}(2))$ is an associative algebra with respect to the multiplication of the formal series.
The associative algebra $TU(\mathfrak{su}(2))$ is called the Taylor extension of $U(\mathfrak{su}(2))$. It is obvious that $TU(\mathfrak{su}(2))$ contains $U(\mathfrak{su}(2))$.

Remark.

The restriction $|n-k|<N$ is important. Consider two series:

$$x_1 := \sum_{k\geq 0} J_+^k\ , \qquad x_2 := \sum_{n\geq 0} J_-^n\ . \tag{30}$$

Their product is reduced to the form

$$x_1 x_2 = \sum_{n,k\geq 0} \Delta_{n,k}(J_0)\, J_-^n\, J_+^k\ , \tag{31}$$

where $\Delta_{n,k}(J_0)$ is not any rational function of J_0, and moreover it is a generalized function of J_0.
The extremal projector $P(\mathfrak{su}(2))$ belongs to the Taylor extension $TU(\mathfrak{su}(2))$. Therefore Löwdin and Shapiro found a solution of the equations $J_+P(\mathfrak{su}(2)) = P(\mathfrak{su}(2))J_- = 0$ not in the space $U(\mathfrak{su}(2))$, but in its extension $TU(\mathfrak{su}(2))$.

Later Shapiro tried to generalized the obtained formula for $P(\mathfrak{su}(2))$ to the case of $\mathfrak{su}(3)$ $(\mathfrak{u}(3))$. The Lie algebra $\mathfrak{u}(3)$ is generated by 9 elements e_{ik} $(i, k = 1, 2, 3)$ with the relations:

$$[e_{ij},\, e_{kl}] \,=\, \delta_{jk}\, e_{il} - \delta_{il}\, e_{kj}\;, \qquad e_{ij}^{*} \,=\, e_{ij}\;. \tag{32}$$

Shapiro considered the following ansatz for $P(\mathfrak{su}(3))$:

$$P(\mathfrak{su}(3)) := \sum_{n_i, m_i \geq 0} C_{n_i, m_i}(e_{11}, e_{22}, e_{33})\, e_{21}^{n_1} e_{31}^{n_2} e_{32}^{n_3} e_{12}^{m_1} e_{13}^{m_2} e_{23}^{m_3} \tag{33}$$

and he used the equations $(P := P(\mathfrak{su}(3)))$:

$$e_{ij}P = Pe_{ji} = 0 \;\; (i < j), \qquad [e_{ii}, P] = 0 \;\; (i = 1, 2, 3). \tag{34}$$

From the last equations it follows that $n_1 + n_2 = m_1 + m_2$, $n_2 + n_3 = m_2 + m_3$. Under these conditions the expression (**??**) belongs to $TU(\mathfrak{su}(3))$. A system of equations for the coefficients $C_{n_i, m_i}(e_{11}, e_{22}, e_{33})$ was found too much complicated, and Shapiro failed to solve this system.

In 1968 R.M. Asherova and Yu.F. Smirnov (Nucl.Phys., v.B4, 399) made the first important step in order to obtain an explicit formula of the extremal projector for $\mathfrak{u}(3)$. They proposed to act with $P(\mathfrak{su}(3))$ described by the Shapiro ansatz (**??**) on the extremal projector of $\mathfrak{su}(2)$ generated by the elements e_{23}, e_{32}, $e_{22}-e_{33}$,

$$\begin{aligned} P_{23} &= \sum_{n\geq 0} \frac{(-1)^n}{n!}\,\varphi_n(e_{22}-e_{33})\,e_{32}^n e_{23}^n\ , && (35)\\ \varphi_n(e_{22}-e_{33}) &= \prod_{k=1}^{n}(e_{22}-e_{33}+k+1)^{-1}\ . && (36) \end{aligned}$$

Since $e_{23}P_{23}=0$, therefore we obtained for $P(\mathfrak{su}(3))$:

$$P=\sum_{n_i\geq 0} C_{n_1,n_2,n_3}(e_{11},e_{22},e_{33})\,e_{21}^{n_1}\,e_{31}^{n_2-n_3}\,e_{32}^{n_3}\,e_{12}^{n_1-n_3}\,e_{13}^{n_2}\,P_{23}\ . \qquad (37)$$

In this case the system of equations for the coefficients $C_{n_i}(e_{ii})$ is more simple and it was solved. However the explicit expressions for the coefficients $C_{n_i}(e_{ii})$ are rather complicated.

The next simple idea (V.N.T. Master's thesis, 1969) was to act on the expression (**??**) (from the left side) by the extremal projector of the subalgebra $\mathfrak{su}(2)$ generated by the elements e_{12}, $e_{21}, e_{11} - e_{22}$. As a result we obtain the following simple form for $P(\mathfrak{su}(3))$:

$$P(\mathfrak{su}(3)) = P_{12}\Big(\sum_{n\geq 0} C_n(e_{11} - e_{33})\, e_{31}^n e_{13}^n\Big) P_{23} \ . \tag{38}$$

The final formula is

$$P(\mathfrak{su}(3)) = P_{12} P_{13} P_{23} \ , \tag{39}$$

where

$$P_{ij} \ = \ \sum_{n\geq 0} \frac{(-1)^n}{n!}\, \varphi_n(e_{ii} - e_{jj})\, e_{ji}^n e_{ij}^n \quad (i<j) \ , \tag{40}$$

$$\varphi_n(e_{ii} - e_{jj}) \ = \ \prod_{k=1}^{n} (e_{ii} - e_{jj} + j - i + k)^{-1} \ . \tag{41}$$

As it turned out this formula is key.

Let us rewrite this expression for $P(\mathfrak{su}(3))$ in the terms of the CW basis with Greek indexes, i.e. we replace the root indexes 12, 23, 13 by α, β, $\alpha+\beta$ correspondingly. Moreover we set $h_\alpha := e_{11} - e_{22}$, $h_\beta := e_{22} - e_{33}$, $h_{\alpha+\beta} := e_{11} - e_{33}$. In these terms the extremal projector $P(\mathfrak{su}(3))$ has the form

$$P(\mathfrak{su}(3)) = P_\alpha P_{\alpha+\beta} P_\beta \ , \tag{42}$$

where

$$P_\gamma = \sum_{n\geq 0} \frac{(-1)^n}{n!} \varphi_{\gamma,n} e_{-\gamma}^{m}\, e_{\gamma}^{m} \ , \tag{43}$$

$$\varphi_{\gamma,n} = \prod_{k=1}^{n} \Big(h_\gamma + \rho(\gamma) + \frac{1}{2}(\gamma,\gamma)k \Big)^{-1} \ . \tag{44}$$

Here ρ is the linear function on the positive root system $\Delta_+ := \{\alpha, \alpha+\beta, \beta\}$, such that $\rho(\alpha) = \frac{1}{2}(\alpha,\alpha)$, $\rho(\beta) = \frac{1}{2}(\beta,\beta)$ for the simple roots α and β.
A generalization of the formulas (**??**)–(**??**) to the case of any simple Lie algebra $\mathfrak{g}$ is given the theorem. Let $e_{\pm\gamma}$, h_γ be Cartan-Weyl root vectors normalized by the condition

$$[e_\gamma, e_{-\gamma}] = h_\gamma \ . \tag{45}$$

Theorem

The equations

$$e_{\gamma}P(\mathfrak{g}) = P(\mathfrak{g})e_{-\gamma} = 0 \quad (\forall\, \gamma \in \Delta_{+}(\mathfrak{g}))\ , \qquad P^{2}(\mathfrak{g}) = P(\mathfrak{g}) \tag{46}$$

have a unique nonzero solution in the space $T(\mathfrak{g})$ and this solution has the form

$$P(\mathfrak{g}) = \prod_{\gamma\in\vec{\Delta}_{+}(\mathfrak{g})} P_{\gamma}\ , \tag{47}$$

for any normal ordering sistem $\vec{\Delta}_{+}(\mathfrak{g})$, where the elements P_{γ} are defined by the formulae

$$P_{\gamma} = \sum_{n\geq 0} \frac{(-1)^{n}}{n!}\, \varphi_{\gamma,n} e_{-\gamma}^{m}\, e_{\gamma}^{m}\ , \tag{48}$$

$$\varphi_{\gamma,n} = \prod_{k=1}^{n}\Big(h_{\gamma} + \rho(\gamma) + \frac{1}{2}(\gamma,\gamma)k\Big)^{-1}\ . \tag{49}$$

Here ρ is the linear function on the positive root system Δ_{+}, such that $\rho(\alpha_i) = \frac{1}{2}(\alpha_i,\alpha_i)$ for all simple roots $\alpha_i \in \Pi$.

The extremal projectors of the Lie algebras of rank 2 and the combinatorial theorem play a key role for the proof of this theorem. The extremal projectors of the Lie algebras of rank 2 are given by

$$\begin{aligned}
P_{\alpha}P_{\beta} &= P_{\beta}P_{\alpha} \qquad (A_1\otimes A_1), && (50)\\
P_{\alpha}P_{\alpha+\beta}P_{\beta} &= P_{\beta}P_{\alpha+\beta}P_{\alpha} \qquad (A_2), && (51)\\
P_{\alpha}P_{\alpha+\beta}P_{\alpha+2\beta,\beta} &= P_{\beta}P_{\alpha+2\beta}P_{\alpha+\beta}P_{\alpha} \qquad (B_2), && (52)\\
P_{\alpha}P_{\alpha+\beta}P_{2\alpha+3\beta}P_{\alpha+2\beta}P_{\alpha+3\beta}P_{\beta} &= P_{\beta}P_{\alpha+3\beta}P_{\alpha+2\beta}P_{2\alpha+3\beta}P_{\alpha+\beta}P_{\alpha} \quad (G_2). && (53)
\end{aligned}$$

We can show that these equations are valid not only for $\rho(\alpha)=\frac{1}{2}(\alpha,\alpha)$, $\rho(\beta)=\frac{1}{2}(\beta,\beta)$ but as well for $\rho(\alpha)=x_{\alpha}$, $\rho(\beta)=x_{\beta}$, where x_{α} and x_{β} are arbitrary complex numbers. Taking into account the combinatorial theorem we immediatly obtain the equlities (**??**).