Acta Applicandae Mathematicae (2005) 85: 49–56
DOI: 10.1007/s10440-004-5585-5

Construction of Semigroups with Some Exotic Properties

A. YA. BELOV and I. A. IVANOV
Moscow State University, Leninskiye Gory, Moscow 119992, Russia

Mathematics Subject Classification (2000): 16Pxx.

Key words: semigroups with noninteger Gelfand–Kirillov dimension, nonnilpotent nilsemigroups, nilrings, recursive presented objects.

The following examples of semigroups, groups and rings with a kit of characteristics which used to be considered exotic are well known. Namely, semigroups with noninteger Gelfand–Kirillov dimension, nonnilpotent nilsemigroups and nilrings, finitely generated infinite periodic groups, and others. However, all these properties were originally reached by means of infinite numbers of identities, so the topics, refering to constructing finitelypresented objects, begin to play a greater role. It's necessary to mention an example of finitely presented associative algebra of intermediate growth by V. A. Ufnarovsky, as well as the results of Higman, G. P. Kukin, V. Ya. Belyaev which dealt with the embedding of recursive presented objects (groups, associative algebras, Lie algebras) into finitely presented groups. The following theorem becomes an object of interest.

THEOREM. *There exists a finitely presented semigroup with noninteger Gelfand–Kirillov dimension.*

We construct our semigroup by means of entering the identities. All the variables except a and b may be included in any nontrivial word not more than once, and the number of elements in any nontrivial word is about $n^{1/2}$, where n is the length of the word. All the other variables (except a and b) represent a mechanism for word analysis. Let $H(x)$ be the number of entrances of x variable in the word. If $H(a) > H(b)^2$, then our mechanism confirms it and takes that word as zero. For any word the H function is equal to one or zero for every variable, except a or b. (We will reach this by means of some special relations.) The number of nontrivial words will be $P(n) * n^{1/2}$, where $P(n)$ is some polynomial function. Let us discuss the mechanism in more detail. C, D and W variables are the *edges* of any nontrivial word (let us have two right edges: D and W). This means that in any nontrivial word, no variables, except b and W, can be to the right of D, no variables, except b, can be to the right of W, and no variables, except a, can be the left of C. R is the separator in the words that contain both a and b variables. If a

and b exist in the word, R variable divides it into the "a" part and "b" part. P and Q variables are the *main heads*. These heads exist in the "a" and "b" parts of the word accordingly. Besides, these heads can interact by means of *signals* (variables f and k). We can construct special relations for main heads and these signals, so that the P head could move inside the "a" part of the word from R to C, while Q moves inside the "b" part of the word from R to D. By means of identities we can obtain that the "speed" of Q would be an approximately squared "speed" of P. Now if Q reaches D earlier than P reaches C, then we take the word as zero. Now we begin to define relations. Let us have variables a, b, C, D, W, R, P, Q, k_1, k_2, $t_1 - t_9$, f_1, f_2. First of all, we will define relations of order, so that any nontrivial word will have the following form:

$$a \dots aCa \dots aPa \dots afa \dots aRb \dots bkb \dots bQb \dots bDb \dots bWb \dots b,$$

where $f = f_1$ or f_2 and $k = k_1$ or k_2. Or any of its subwords $ab = ba = 0$, i.e. $a \odot b$ are divided

$$P = t_1 P,$$
$$Q = Qt_2,$$
$$D = Dt_3,$$
$$W = Wt_9,$$
$$t_1 * = * t_1$$

($*$ – any, except P and C)

$$t_2 * = * t_2$$

($*$ – any, except Q and D)

$$t_3 * = * t_3$$

($*$ – any, except D and W)

$$t_9 * = * t_9$$

($*$ – any, except W)

$$* t_1 = 0$$

($*$ – any, except a, C, t_1)

$$t_2 * = 0$$

($*$ – any, except b, t_2, D and W)

$$t_3 * = 0$$

($*$ – any, except b, t_3 and W);

$$t_9* = 0$$

($*$ – any, except b and t_9)

$$C = t_4C,$$

$$t_4* = *t_4$$

($*$ – any, except C)

$$*t_4 = 0$$

($*$ – any, except a), i.e. to the left of P there may be only $a \dots aCa \dots a$, and to the right of Q only $b \dots bDb \dots bWb \dots b$. ($C$, D and W are included to the word only once each.)

$$aQ = Qa = 0,$$
$$Pb = bP = 0,$$
$$f_1a = af_1.$$

k_1, k_2 commute with a, b and R

$$RR = 0,$$
$$aD = aW = 0,$$
$$PQ = QP = 0.$$

Squares of f_1, f_2, k_1, k_2 are also zeros: $f_1^2 = f_2^2 = k_1^2 = k_2^2 = 0$. By means of all these relations we can obtain that in any nontrivial word $a \dots a$ and $b \dots b$ may be separated by R only. There can be f_1, f_2, k_1, k_2, C, P inside $a \dots a$ and Q, D, k_1, k_2 inside $b \dots b$. By the following relations:

$$R = Rt_5$$

t_5 commutes with k_1, k_2 and b

$$t_5* = 0$$

(if $*$ is not k_1, k_2 or b)

$$t_5Q = Q$$

we force the existence of Q in the presence of D, and the fact that between R and Q there can be only k_1, k_2 or b. By means of relations

$$R = t_6R$$

t_6 commutes with k_1, k_2, f_1, f_2 and b

$$*t_6 = 0$$

(if $*$ is not k_1 k_2 f_1 f_2 or a)

$$Pt_6 = P$$

we force the existence of P in the presence of C, and the fact that between R and P there can be only k_1 k_2 f_1 f_2 or a.

$$f_2 f_1 = f_1 f_2 = 0,$$
$$k_1 k_2 = k_2 k_1 = 0,$$
$$kf = 0$$

(k is k_1 or k_2 and f is f_1 or f_2). Since, in any nontrivial word we can find f_1 or f_2 and k_1 or k_2 (f_1 commutes with a, k_1 k_2 commute with $a \odot b$).

$$P = Pt_7,$$
$$t_7 a = at_7,$$
$$t_7 f_1 = f_1,$$
$$t_7 f_2 = f_2,$$
$$t_7 * = 0$$

($*$ – any, except f_1 and f_2). By means of these relations we can obtain the existence f_1 or f_2 after several a (to the right of R). Since, any nontrivial word will have the following form:

$$a \dots aCa \dots aPa \dots afa \dots aRb \dots bkb \dots bQb \dots bDb \dots bWb \dots b,$$

where $f = f_1$ or f_2 and $k = k_1$ or k_2. Or any of its subwords. Below we will consider, how many words without some heads or separators can exist. Now let us have the main case. Indeed, there can be only a and C to the left of P, only a to the left of C. There can be only b and D to the right of Q, only b to the right of D. Besides, if P exists, then f exists to the right side (f is f_1 or f_2), if both a and b exist in one word, then R would be a separator. There are no more separators except R. The variables of type t will take the word as zero, if it appears in a nonintended area (e.g., t_5 within P and Q). The number of the words where an a or b variable is absent is about n^3. Now we will discuss the general mechanism. Let P and Q exist in the word (and f is forced). Let us define additional relations.

$$f_2 = f_2 t_8,$$
$$t_8 a = at_8,$$
$$t_8 R = Rt_8,$$
$$t_8 Q = 0.$$

By means of f_2 we force k_1 or k_2 (otherwise t_8 will reach Q and make word equal to zero).

$$aPf_1 = Paf_2k_1$$

(f_1 moves P in the $a \dots a$ massive, and turns into f_2k_1).

$$k_1Qb = k_2bQ$$

(k_1 commutes with a, b, R and moves to Q, pushing Q to the right in $b \dots b$ massive and turns into k_2).

$$f_2k_2a = af_2k_1$$

(k_1 returns to f_2, moves f_2 for one point to the left and turns into k_2). Then k_2 moves again to f_2, etc. Since, the $k_1 - k_2$ mechanism moves f_2 to the right as much as Q does.

$$f_2Rk_2 = f_1R.$$

(When f_2 returns to R, it turns into f_1 again.) Then f_1 returns to P again (f_1 commutes with a) and the cycle repeats. Since, the mechanism $f_1 - f_2 - k_1 - k_2$ moves P for one point to the left, and Q to the right for the distance from R to P. Q is accelerating, P moves with constant speed. If P reaches C, when the process stops, $QD = 0$ (i.e. if the distance between C and P is too small (regarding to Q and D), then we take the word as zero. Let us have an arbitrary word, then P and Q are situated in a and b parts of the word. Let us consider the reverse work of the process of P and Q moving. We can move P and Q to R, until P or Q meets R. By means of these relations:

$$aaRQ = 0,$$
$$PRbb = 0$$

we obtain that P and Q will be situated at an equal distance from R. Since, any nontrivial word will be equivalent to the word containing P, R and Q situated close by. Now let us consider the straight work of the process. There are few words without C or D. Let C and D be the word. Since, the length of the b part of the word is an almost squared length of the a part. Now let us consider the situation when some heads or the separator do not exist. If the word does not include b, there can be only C, P, f_2 or f_1, k_1 or k_2, R in the end of the word and some variables of t-type. f_1, k_1, k_2 and t-type variables commute with a, so we can count the number of nontrivial words in this case by considering the location of C, P and f_2. So the number of words of a length less than n is about n^4. If the word includes b and does not include a, we can consider only the locations of Q, D and W. So the number of words is about n^4. Let us have both a and b variables in the word. Than R is forced. If D, Q and W do not exist, we can consider only the locations of

R, C, P and f_2 variables. So the number of words is about n^5. Similarly, if C, P and f_2 do not exist, we can consider the locations of Q, D and W variables. (In this case, if f_2 exists, the mechanism makes "part of the work" and f_2 turns into f_1.) If both P and Q head variables exist in the word, our mechanism, considered above, begins to work. The reverse work of the mechanism moves both P and Q head variables to R close enough (otherwise the word will be zero). So, if D and W are not included in the word, we can consider only the locations of R and C. So the number of words is about n^4. In the main case, all edges are in the word. The distance between R and D is more than the squared distance between C and R. W can be located everywhere to the right of D. So the number of the nontrivial words with length n in this case is about

$$\sum_{k=1}^{n} \sum_{i=1}^{\min(\sqrt{n-k},k)} (n-i^2-k)(k-i)^2,$$

i.e. about $n^4 * \sqrt{n}$. So, the number of the words with length less than n is about $n^5 * \sqrt{n}$. If some of the edges do not exist, then we have the number of such words less than $n^5 * \sqrt{n}$. Since we have about $n^5 * \sqrt{n}$ various words in our semigroup. The proof is finished. Now let us improve our construction.

THEOREM. *For any rational $\alpha > 5$ there exists a finitely presented semigroup with Gelfand–Kirillov dimension α.*

We now consider the *main heads* as the main variable (e.g., P) with the kit of auxiliary variables (e.g., k_1, k_2, f_1, f_2), providing the connection with another head or heads. Let us have s various *main heads* such as P or Q instead of two heads. Every head interacts with the head of the next number and "ignores" any other head. This means that any part of the first head commutes with any part of the second head (i.e. heads are simplyds not notice each other). By means of relations like those considered above, we can obtain that for any $k < n$ the head with number $k+1$ would have an acceleration regarding the head with number k. Since, if head with number 2 has a speed like t, then the head with number s has a speed like t^{s-1}. Now we can construct a semigroup with any large enough rational Gelfand–Kirillov dimension. Let p/q be the fraction part of number α that we want to construct. Let s be equal to $q+1$, then $p < s-1$. Let P be the head with speed t^p and Q be the head with speed t^q. By means of relations like those considered above (for a head relationship with the edges of the word) we can obtain that if the Q head reaches its edge (e.g., D) earlier then P does, then the word would be zero. Since, the length of the a part of the word would be about $n^{p/q}$. Let us define relations consecutively. First of all, let us take the relations defined above for a semigroup with dimension 5.5. Let us mark P and Q head variables along with operating variables such as f_1, f_2, and P^1, Q^1, etc. Let us call them *variables of the first step*. We are deleting the relation $Q^1D = 0$. Now let us define the variables of the second step. Let us take the variables P^2 and others and define the relations as for P^1, but now in these relations we are considering variables of the first step except Q^1 as a or b. So the second step variables will ignore the first step variables. Let us make the relations between P^2 and Q^1 as the relations between Q^1 and P^1 (exactly in this order). These are the relations of the main mechanism of acceleration. (We are defining ${f_1}^2$, ${k_1}^2$, and other variables for it.) So, the speed of P^2 is about t^2. (The speed of P^1 is constant, the speed of Q^1 is about t – we have acceleration.) Now we are defining the relations of the third step. Q^3 will "ignore"everything except P^2. The speed of Q^3 will be t^3. So we are defining a new head for step $k + 1$. That head ignores everything except the last-defined head of the previous step and interconnecting with this head. For this interconnection we are defining the relations of step $k + 1$. So, if the fraction part of α is equal to p/q, we are defining up to $p - 1$ steps. Let P^{p-1} be the last head, operating within its side of the separator R. (We will place the next step heads at the other side of the separator R. It will requive small changes in the main mechanism of acceleration.) After $q - 1$ steps, we define one of the following relations:

$$Q^q D = 0$$

or

$$P^q C = 0$$

(this depends on the parity). Since we will have the dimension with the fraction part as being equal to p/q, the proof is finished. To construct a recursive dimension we need to improve our conception of heads and transmitting signals. Let us have two head variables, E_1 and E_2. We can consider this combination as the *complex head*. This complex head can receive the signals like a simple head, but it has some advantages. The point is in the changing distance between the edges of this complex head. Let us call the distance between the edges of that head the *power of the head*. By means of some relations we can obtain that any complex head with power more then zero can force the existence of the other complex head to the left. Besides, we can construct a mechanism for the head relationship that would control the connection between the powers of that heads. Since we can obtain the existence of the chain of complex heads, with a zero-powered last head in the chain, the power of every head in that chain is about a squared power of the next head in the chain. Let us enable every head to include *the buffer zone*. That zone is situated outside the edges of the head and has the same power as that head

$$E_0 a \ldots a E_1 a \ldots a E_2.$$

So, the $E_0 a \ldots a E_1$ is the buffer zone. By means of the simple mechanism in the relations, we can obtain that the power of the zone will be the same as the head power. The buffer zone is not a necessary addition to the head. Besides, let us enable all to include a *manager symbol* that shows the existence of the buffer zone. To define the manager symbols we construct a special mechanism. The law of the distribution of manager symbols depends on our recursive number α and on the algorithm of definition of the digits of the number α. Every head includes a part of that algorithm. Let the manager symbols be defined. Let us consider the sum of the powers of the heads. Some of them have buffer zones. By means of manager symbols we can approximate the number α by the sum of the finite number of powers of the heads.