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TRISECANT LEMMA FOR NONEQUIDIMENSIONAL VARIETIES

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Abstract. Let X be an irreducible projective variety over an algebraically closed field of characteristic zero. For $r \geq 3$, if every $(r-2)$-plane $\overline{x_1, \ldots, x_{r-1}}$, where the x_i are generic points, also meets X in a point x_r different from $x_1, \ldots, x_{r-1}$, then X is contained in a linear subspace L such that $\operatorname{codim}_L X \leq r-2$. In this paper, our purpose is to present another derivation of this result for $r = 3$ and then to introduce a generalization to nonequidimensional varieties. For the sake of clarity, we shall reformulate our problem as follows. Let Z be an equidimensional variety (maybe singular and/or reducible) of dimension n, other than a linear space, embedded into $\mathbb{P}^r$, where $r \geq n+1$. The variety of trisecant lines of Z, say $V_{1,3}(Z)$, has dimension strictly less than $2n$, unless Z is included in an $(n+1)$-dimensional linear space and has degree at least 3, in which case $\dim V_{1,3}(Z) = 2n$. This also implies that if $\dim V_{1,3}(Z) = 2n$, then Z can be embedded in $\mathbb{P}^{n+1}$. Then we inquire the more general case, where Z is not required to be equidimensional. In that case, let Z be a possibly singular variety of dimension n, which may be neither irreducible nor equidimensional, embedded into $\mathbb{P}^r$, where $r \geq n+1$, and let Y be a proper subvariety of dimension $k \geq 1$. Consider now S being a component of maximal dimension of the closure of $\{l \in \mathbb{G}(1,r) \mid \exists p \in Y,\ q_1, q_2 \in Z \backslash Y,\ q_1, q_2, p \in l\}$. We show that S has dimension strictly less than $n+k$, unless the union of lines in S has dimension $n+1$, in which case $\dim S = n+k$. In the latter case, if the dimension of the space is strictly greater than $n+1$, then the union of lines in S cannot cover the whole space. This is the main result of our paper. We also introduce some examples showing that our bound is strict.

1. Introduction

The classic trisecant lemma states that if X is an integral curve in $\mathbb{P}^3$, then the variety of trisecants has dimension 1, unless the curve is planar and has degree at least 3, in which case the variety of trisecants has dimension 2. Several generalizations of this lemma have been considered [1, 2, 4, 7, 9]. In [7], the case of an integral curve embedded in $\mathbb{P}^3$ is further investigated, leading to a result on the planar sections of such a curve. On the other hand, in [9], the case of higher dimensional varieties, possibly reducible, is inquired. For our concern, the main result of [9] is that if m is the dimension of the variety, then the union of a family of $(m+2)$-secant lines has dimension at most $m+1$. A further generalization of this result is given in [1, 2, 4]. In this latter case, the setting is the following. Let X be an irreducible projective variety over an algebraically closed field of characteristic zero. For $r \geq 3$, if every $(r-2)$-plane $\overline{x_1, \ldots, x_{r-1}}$, where the x_i are generic points, also meets X in a point x_r different from $x_1, \ldots, x_{r-1}$, then X is contained in a linear subspace L, with $\operatorname{codim}_L X \leq r-2$.

In this paper, our purpose is first to present another derivation of this result for $r = 3$ and then to introduce a generalization to nonequidimensional varieties. For the sake of clarity, we shall reformulate our first problem as follows. Let Z be an equidimensional variety (maybe singular and/or reducible) of dimension n, other than a linear space, embedded into $\mathbb{P}^r$, where $r \geq n+1$. The variety of trisecant lines of Z, say $V_{1,3}(Z)$, has dimension strictly less than $2n$, unless Z is included in an $(n+1)$-dimensional linear space and has degree at least 3, in which case $\dim V_{1,3}(Z) = 2n$. This also implies that if $\dim V_{1,3}(Z) = 2n$, then Z can be embedded in $\mathbb{P}^{n+1}$.

Then we inquire the more general case, where Z is not required to be equidimensional. In that case, let Z be a possibly singular variety of dimension n, which may be neither irreducible nor equidimensional, embedded into $\mathbb{P}^r$, where $r \geq n+1$, and let Y be a proper subvariety of dimension $k \geq 1$. Consider now S being a component of maximal dimension of the closure of $\{l \in \mathbb{G}(1,r) \mid \exists p \in Y,\ q_1, q_2 \in Z \backslash Y,\ q_1, q_2, p \in l\}$.

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We show that S has dimension strictly less than $n+k$, unless the union of lines in S has dimension $n+1$, in which case $\dim S = n+k$. In the latter case, if the dimension of the space is strictly greater than $n+1$, then the union of lines in S cannot cover the whole space. This is the main result of our work. We also introduce some examples showing that our bound is strict.

The methods we use to prove these results are purely algebraic and are valid over any algebraically closed field of characteristic zero. Our reasoning consists basically in inquiring first the local restrictions on the tangent spaces for the trisecant lines variety being of full dimension. Then the global result is deduced using the so-called Terracini's lemma [11].

The paper is organized as follows. First we recall some standard material in order to fix terminology and notations in Sec. 2. Then we come to our results in Sec. 3. More precisely, in Sec. 3.2, the case of equidimensional varieties is investigated, while in Sec. 3.3 we deal with the more general case.

2. Notations and Background

In this section, we recall some standard material on incident varieties, which will be used in the sequel. The ground field is always assumed to be of characteristic zero.

2.1. Variety of Incident Lines. Let $\mathbb{G}(1,n) = G(2,n+1)$ be the Grassmannian of lines included in $\mathbb{P}^n$. Recall that $\mathbb{G}(1,n)$ can be canonically embedded in $\mathbb{P}^{N_1}$, where $N_1 = \binom{2}{n+1} - 1$, by the Plücker embedding and that $\dim \mathbb{G}(1,n) = 2n-2$. Hence a line in $\mathbb{P}^n$ can be regarded as a point in $\mathbb{P}^{N_1}$ satisfying the so-called Plücker relations. These relations are quadratic equations that generate a homogeneous ideal, say $I_{\mathbb{G}(1,n)}$, defining $\mathbb{G}(1,n)$ as a closed subvariety of $\mathbb{P}^{N_1}$. Similarly, the Grassmannian $\mathbb{G}(k,n)$ gives a parametrization of the k-dimensional linear subspaces of $\mathbb{P}^n$. As for $\mathbb{G}(1,n)$, the Grassmannian $\mathbb{G}(k,n)$ can be embedded into the projective space $\mathbb{P}^{N_k}$, where $N_k = \binom{k+1}{n+1} - 1$. Therefore, for a k-dimensional linear subspace K of $\mathbb{P}^n$, we shall write $[K]$ for the corresponding projective point in $\mathbb{P}^{N_k}$. The line passing through some points x and y will be denoted $\overline{xy}$.

Definition. Let $X \subset \mathbb{P}^n$ be an irreducible variety. We define the following variety of incident lines:

$$\Delta(X) = \{l \in \mathbb{G}(1,n) \mid l \cap X \neq \varnothing\}.$$

The codimension c of X and the dimension of $\Delta(X)$ are related by the following lemma.

Lemma 1. *Let $X \subset \mathbb{P}^n$ be an irreducible closed variety of codimension $c \geq 2$. Then $\Delta(X)$ is an irreducible subvariety of $\mathbb{G}(1,n)$ of dimension $2n-1-c$.*

Proof. Consider the incidence variety

$$\Sigma = \{(l,p) \in \mathbb{G}(1,n) \times X \mid p \in l\} \subset \Delta(X) \times X$$

endowed with the canonical projections $\pi_1 \colon \Sigma \to \Delta(X)$ and $\pi_2 \colon \Sigma \to X$. The generic fiber of π_1 is finite (otherwise it is clear that $X = \mathbb{P}^n$). Thus, $\dim \Sigma = \dim \Delta(X)$. For all $p \in X$, the fiber $\pi_2^{-1}(p)$ is isomorphic to $\mathbb{P}^{n-1}$ and has dimension $n-1$. Therefore, Σ is irreducible and has dimension $n-c+n-1 = 2n-c-1$, as shown in [5,10]. Since π_1 is surjective and continuous (in Zariski topology), then $\Delta(X)$ is also irreducible and has dimension $2n-c-1$. $\square$

The following simple result will be useful in the sequel.

Lemma 2. *Let X_1 and X_2 be two irreducible closed varieties in $\mathbb{P}^n$ of codimension greater than or equal to 2. Then $\Delta(X_1) \not\subset \Delta(X_2)$ unless $X_1 \subset X_2$.*

Proof. Assume that $\Delta(X_1) \subset \Delta(X_2)$ and $X_1 \not\subset X_2$. Consider a point $p \in X_1 \setminus X_2$ and a hyperplane H not passing through p. Consider the projection $\pi \colon \mathbb{P}^n \setminus \{p\} \to H$, $q \mapsto \overline{qp} \cap H$ that maps a point $q \in \mathbb{P}^n \setminus \{p\}$ to the point of intersection of the line $\overline{qp}$ with the hyperplane H. The projection is surjective, and so is $\pi|_{X_2}$, because $\Delta(X_1) \subset \Delta(X_2)$. Thus, $\dim X_2 \geq n-1$, which is impossible, because $\operatorname{codim} X_i \geq 2$ for each i. $\square$

2.2. Join Varieties. Consider $m < n$ closed irreducible varieties $\{Y_i\}_{i=1,\dots,m}$ embedded in $\mathbb{P}^n$ with codimensions $c_i \geq 2$. Consider the *join variety*

$$J = J(Y_1, \dots, Y_m) = \Delta(Y_1) \cap \dots \cap \Delta(Y_m)$$

included in $\mathbb{G}(1,n)$. We assume that $\sum\limits_{i=1,\dots,m} c_i \leq 2n - 2 + m$, so that J is not empty. We shall first determine the irreducible components of J.

Let U be the open subset of $Y_1 \times \dots \times Y_m$ defined by

$$\{(p_1, \dots, p_m) \in Y_1 \times \dots \times Y_m \mid \exists i \neq j \colon p_i \neq p_j\}.$$

Let V be the locally closed set made of the m-tuples in U whose points are collinear. Let $s \colon V \to \mathbb{G}(1,n)$ be the morphism that maps an m-tuple of aligned points to the line they generate. Let $S \subset \mathbb{G}(1,n)$ be the closure of the image of s.

First, let us look at the irreducible components of S. These components could be classified in several classes according to the number of distinct points in the m-tuples that generate them. For example, consider the case where $m = 3$. The locally closed subset of $Y_1 \times Y_2 \times Y_3$, made of triplets of three distinct and collinear points, generates one component of S. Now if Y_{12} is an irreducible component of $Y_1 \cap Y_2$ not contained in Y_3, then the lines generated by a point of $Y_{12} \setminus Y_3$ and another point in Y_3 form also an irreducible component of S. Also let Z be an irreducible component of $Y_1 \cap Y_2 \cap Y_3$; then the lines generated by a point of Z and another point in Y_1 are the intersection of the secant variety of Y_1 with $\Delta(Z)$, and form an irreducible component of S too. In the general case, the following lemma will suffice for our purpose.

Lemma 3. *The irreducible components of J are*:

(1) $\Delta(Z)$*, where Z runs over all irreducible components of* $Y_1 \cap \dots \cap Y_m$;
(2) *the irreducible components of S that are not included in any component of the form $\Delta(Z)$.*

Proof. These sets are all irreducible closed subsets of J. There is a finite number of such sets, and their union covers J. Thus, the irreducible components of J are certainly some of these sets.

Suppose that for some irreducible component Z of $Y_1 \cap \dots \cap Y_m$ we have $\Delta(Z) \subset S$. Let us proceed similarly to Lemma 2. Consider a point $p \in Z$ and a hyperplane H not passing through p. Consider the projection $\pi \colon \mathbb{P}^n \setminus \{p\} \to H$, $q \mapsto \overline{qp} \cap H$ that maps a point $q \in \mathbb{P}^n \setminus \{p\}$ to the point of intersection of the line $\overline{qp}$ with the hyperplane H. The projection is obviously surjective. Since $\Delta(Z) \subset S$, each line l meeting p is the limit of lines of the form $\overline{p'q}$, where $p' \in Z$ and q is some other point of $(Y_1 \cup \dots \cup Y_m)$. Choosing the points p' tending to p, we see that l is the limit of lines $\overline{pq}$. It follows that the projection of $(Y_1 \cup \dots \cup Y_m)$ is dense in H. But this is impossible since $\operatorname{codim} Y_i \geq 2$ for each i. Thus, $\Delta(Z) \not\subset S$.

Now by Lemma 2, $\Delta(Z_1) \not\subset \Delta(Z_2)$ for any two irreducible components Z_1 and Z_2 of $Y_1 \cap \dots \cap Y_m$. Since the set of lines meeting an irreducible variety is irreducible, $\Delta(Z)$ is a maximal irreducible closed subset of J for every irreducible component Z of $Y_1 \cap \dots \cap Y_m$.

Every irreducible component S_1 of S that is not included in any component of the form $\Delta(Z)$ is also a maximal irreducible closed subset of J. $\square$

For simplicity, we shall call the irreducible components of S *joining components* of J and components of the form $\Delta(Z)$ for some irreducible component Z of $Y_1 \cap \dots \cap Y_m$ *intersection components*.

We conclude this section by quoting Terracini's lemma, in the form we shall use later. For this purpose and throughout the paper, we use the following notations. If X is a projective subvariety of $\mathbb{P}^n$, then we shall write $T_p(X)$ for the projectively embedded tangent space of X at p. The Zariski tangent space is denoted $\Theta_p(X)$. Let CX be the affine cone over X; then $T_p(X)$ is the projective space of one-dimensional subspaces of $\Theta_q(CX)$, where $q \in \mathbb{A}^{n+1}$ is any point lying over p. Hence for a morphism f between two projective varieties X and Y, which can also be viewed as a morphism between CX and CY, the differential $df_p \colon T_p(X) \setminus \mathbb{P}(\ker(\phi)) \to T_{f(p)}(Y)$ is induced by the differential ϕ between the Zariski tangent spaces $\Theta_q(CX)$ and $\Theta_{f(q)}(CY)$. For simplicity, we shall write $df_p\colon T_p(X) \to T_{f(p)}(Y)$, while it is understood that df_p might be defined on a proper subset of $T_p(X)$.

Lemma 4 (Terracini's lemma)**.** *Let X and Y be two irreducible projective varieties embedded in $\mathbb{P}^n$ over an algebraically closed field of characteristic zero. Let $W(X, Y)$ be the union of the lines in $J(X, Y)$. Let z be a point in $W(X, Y)$ lying neither on X nor on Y. Then the tangent space of $W(X, Y)$ at z is given by the following equality*:

$$T_z(W(X, Y)) = \langle T_x(X), T_y(Y)\rangle,$$

where $(x, y) \in X \times Y$, such that $z \in \langle x, y\rangle = \overline{xy}$ and $\langle\ \rangle$ denotes the linear span.

A slightly more general statement and its proof can be found in [11].

3. Generalizations of the Trisecant Lemma

In this section, we shall introduce two generalizations of the trisecant lemma. The first one is about equidimensional varieties, while the second one deals with a more general situation.

In terms of join varieties, the classical trisecant lemma and the generalizations we introduce are related to join components. A similar treatment of intersection components is easy to give and is summarized in the following lemma, immediately deduced from Lemma 2.

Lemma 5. *Let Y_1 and Y_2 be two distinct irreducible varieties embedded in some projective space. Let Y be a third irreducible variety. $\Delta(Y)$ cannot contain any intersection component $\Delta(Z)$ of $J(Y_1, Y_2) = \Delta(Y_1) \cap \Delta(Y_2)$, unless $Z \subset Y$.*

Before we proceed, we shall prove some results, useful in the sequel.

3.1. Preliminary Properties. The technique of the following proposition is typical for this paper. The proposition can be viewed as a generalization of a well-known result of Samuel [6, p. 312], which deals with smooth curves.

Proposition 1. *Let X be an irreducible closed subvariety of $\mathbb{P}^n$ of dimension k. If there exists $L \in \mathbb{G}(k-1, n)$ such that for all points $p \in U_0$, where U_0 is a dense open set of X and $L \subset T_p(X)$, then X is a k-dimensional linear space containing L.*

Proof. Let $\mathcal{T}X$ be the closure of

$$\{[T_p(X)] \mid p \in X,\ p \text{ regular}\}$$

in $\mathbb{G}(k, n)$. $\mathcal{T}X$ is the closure of the image of a dense open set of X by the Gauss map. Therefore, $\mathcal{T}X$ is irreducible. Consider the rational map $X \dashrightarrow \mathbb{G}(k, n)$, $p \mapsto p \vee L$, where $\vee$ is the join operator [3] equivalent to the classical exterior product (as in [3], the departure from the classical notation is amply justified by the geometric meaning of the operator). Let σ_L be the subvariety of $\mathbb{G}(k, n)$ made of the linear spaces that contain L. Then $\dim \sigma_L = n - k$.

Let U be the open set of X made of the regular points of U_0 that do not lie on L. Consider the morphism $f\colon U \to \sigma_L$, $p \mapsto p \vee L$. For each $p \in U$, $f(p)$ is simply the tangent space of X at p. Therefore, the image of f is dense in $\mathcal{T}X$.

Since the ground field is assumed in this article to have characteristic zero, there exists a dense open set V of X such that for any point p in V, the differential df_p is surjective [6, p. 271].

This differential is simply $df_p\colon T_p(X) \to T_{f(p)}(\mathcal{T}X)$, $a \mapsto a \vee L$. Therefore, df_p is constant over $T_p(X) \setminus L$ and takes the value $[T_p(X)] = df_p(p)$. Thus, $\dim(\mathcal{T}X) = 0$. Since $\mathcal{T}X$ is irreducible, it is a single point corresponding to a k-dimensional linear space, say T, containing L. Finally, $X \subset T$, $\dim X = k$, and X is closed. Therefore, we have $X = T$. $\square$

Note that this fact does not hold in positive characteristic, as the following example shows. Consider the curve in $\mathbb{P}^3$ over a field K of characteristic p, defined by the ideal

$$\langle y^p - zt^{p-1},\ x^p - yt^{p-1}\rangle \subset K[x, y, z, t]$$

with $t = 0$ being the plane at infinity. The tangent space at (x_0, y_0, z_0, t_0) is given by the following system of linear equations:

$$\{t_0^{p-1}z + (p-1)z_0t_0^{p-2}t = 0,\ t_0^{p-1}y + (p-1)y_0t_0^{p-2}t = 0\}.$$

Any two tangent spaces are parallel, and therefore all of them contain the same point at infinity. However, the curve is not a line. Note that the point $(0, 0, 1, 0)$ is a singular point of the curve.

The next proposition is used throughout the paper several times. The underlying idea is as follows. Let L be a k-dimensional linear space. If the tangent space to an irreducible variety at a generic point always spans together with L a $(k+1)$-dimensional linear space, then the variety itself can be included into a $(k+1)$-dimensional linear space containing L.

Proposition 2. *Let X be an irreducible closed subset of $\mathbb{P}^n$ with $\dim X = r$. If there exists $L \in \mathbb{G}(k, n)$ such that for all points $p \in U_0$, where U_0 is a dense open subset of X, $\dim(L \cap T_p(X)) \geq r - 1$, then X is included in a $(k+1)$-dimensional linear space containing L.*

Proof. If $X \subset L$, then there is nothing to prove. Therefore, let us assume that $X \not\subset L$. Let $\sigma_L \subset \mathbb{G}(k+1, n)$ be the set of $(k+1)$-dimensional linear spaces that contain L. Consider the rational map $f\colon X \dashrightarrow \sigma_L$, $p \mapsto p \vee L$. This map is defined over the open set U of regular points in $(X \setminus L) \cap U_0$. Each such point is mapped to the $(k+1)$-dimensional space generated by p and L. Since $\dim(T_p(X) \cap L) = r - 1$, we have the inclusion $T_p(X) \subset p \vee L = f(p)$ for $p \in U$. Let Y be the closure of $f(U)$ in σ_L. Thus, Y is irreducible.

Since the ground field is assumed to have characteristic zero, there exists a dense open set V of X such that for any point p in V, the differential df_p is surjective [6, p. 271].

This differential is simply $df_p\colon T_p(X) \to T_{f(p)}(Y)$, $a \mapsto a \vee L$. Since $T_p(X) \subset p \vee L$, df_p is constant over $T_p(X) \setminus L$ and takes the value $p \vee L = df_p(p)$. Thus, $\dim Y = 0$. Since Y is irreducible, Y is a single point corresponding to a $(k+1)$-dimensional linear space, say K, containing L. Therefore, $X \subset K$. ☐

This proposition does not hold in positive characteristic. Indeed, over a field of characteristic p, for the curve in $\mathbb{P}^3$ defined by the ideal $\langle yt^{p-1} - x^p,\ zt^{p^2-1} - x^{p^2}\rangle$, all the tangent lines are parallel and, therefore, intersect in some point at infinity. But the curve is not a line.

Before we come to investigate our initial question, let us first show, in the case of two varieties embedded in $\mathbb{P}^n$ with $n \geq 3$, that the join has necessarily a unique joining component, which has the required dimension, namely $2n - (c_1 + c_2)$.

Lemma 6. *Let Y_1 and Y_2 be two distinct irreducible varieties embedded in $\mathbb{P}^n$. Let $c_i \geq 2$ be the codimension of Y_i. Then the join $J = J(Y_1, Y_2)$ has a unique joining component S, whose dimension is $2n - (c_1 + c_2)$.*

Proof. Let $\Delta = \{(y_1, y_2) \in Y_1 \times Y_2 \mid y_1 = y_2\}$. Let U be the open subset of $Y_1 \times Y_2$ defined as $U = (Y_1 \times Y_2) \setminus \Delta$. Let $s\colon U \to \mathbb{G}(1, n)$, $(p, q) \mapsto \overline{pq}$ be the morphism that maps an element of U to the line it generates. Let S be the closure of $s(U)$ in $\mathbb{G}(1, n)$. Since U is irreducible, so is S. It is, therefore, the unique joining component of J. The general fiber is finite. Thus, $\dim S = \dim U = \dim(Y_1 \times Y_2) = 2n - (c_1 + c_2)$. ☐

Eventually, we also have the following lemma, which will be useful in the sequel.

Lemma 7. *Let Y_1 and Y_2 be two irreducible varieties embedded in $\mathbb{P}^n$, with dimensions d_1 and d_2 both smaller than or equal to $n - 2$. Let S be the unique joining component of $J = J(Y_1, Y_2)$. Then $\dim(S) = s = d_1 + d_2$.*

The union of the lines in S is an irreducible variety of dimension strictly greater than $\max(d_1, d_2)$. For a generic point p in Y_i, the dimension of the variety of lines in S passing through p is d_{3-i}.

Moreover, if there exists an irreducible variety Y of dimension $d \leq \max(d_1, d_2)$ such that $S \subset \Delta(Y)$, then $d = \max(d_1, d_2)$ and for a generic point p in Y the dimension of the variety of lines in S passing through p is $\min(d_1, d_2)$.

Proof. We shall assume, without loss of generality, that $d_1 \geq d_2$.

Step 1. Consider first the incidence variety $\Sigma = \{(l, p) \in S \times \mathbb{P}^n \mid p \in l\}$ endowed with the canonical projections $\pi_1 \colon \Sigma \to S$ and $\pi_2 \colon \Sigma \to \mathbb{P}^n$. For all $l \in S$, the fiber $\pi_1^{-1}(l)$ is irreducible and has dimension 1 and S is irreducible. Thus, Σ is irreducible and $\dim \Sigma = \dim S + 1 = s + 1$. Let

$$W = \pi_2(\pi_1^{-1}(S)) = \bigcup_{l \in S} l.$$

Then W is irreducible, since Σ is irreducible. Since $Y_i \subset W$ for each i, we see that $\dim W \geq \max(d_1, d_2)$. Furthermore, the generic fiber of π_2 has dimension less than or equal to d_2. Indeed, the fiber at a generic point p is included in $\{(\overline{qp}, p) \mid q \in Y_2\}$. Thus, $\dim W > \max(d_1, d_2)$.

Step 2. For $p \in Y_1$, consider the open set $U = Y_2 \setminus \{p\}$ and the morphism $f \colon U \to S$, $q \mapsto \overline{pq}$. Since U is irreducible, so is $f(U)$. For a generic line l in $f(U)$, the fiber $f^{-1}(l)$ is finite (otherwise Y_2 is a cone with vertex p, which is impossible for a generic $p \in Y_1$). Therefore, $\dim f(U) = \dim Y_2$. A similar conclusion is valid for a generic point of Y_2. Therefore, the dimension of the variety of lines in S passing through a general point in Y_i is d_{3-i}.

Step 3. For a point $p \in Y$, let X_p be the variety of lines in S passing through p. Let Z be the subvariety of Y defined as the set of points for which X_p is not empty. Then $S \subset \Delta(Z)$.

Let us show that Z is irreducible. Let $Z = E \cup F$, where E and F are closed subsets of Z. Denote by S_1 and S_2 the unique joining components of $J(E, Y_2)$ and $J(F, Y_2)$, respectively. Then $\dim S_1 = \dim E + d_2$ and $\dim S_2 = \dim F + d_2$. Moreover, $S \subset S_1 \cup S_2$. Therefore, $\max(\dim E + d_2, \dim F + d_2)) \geq d_1 + d_2$, so that $\max(\dim E, \dim F) \geq d_1$. However, $\dim Z \leq \dim Y \leq d_1$. We conclude that either $E = Z$ or $F = Z$, and $\dim Z = d_1$. Thus, $Z = Y_1$ and $\dim Y = d_1$.

Let S' be the unique joining component of $J(Z, Y_2)$. Then we have $S \subset S'$. But $\dim S = \dim S'$ and both varieties are irreducible closed varieties. Thus, $S = S'$. By a similar argument as in step 2, we get that for a generic point p in Y, the dimension of X_p is $d_2 = \min(d_1, d_2)$. $\square$

3.2. Equidimensional Varieties. We are in a position to present our derivation of the general trisecant lemma valid for equidimensional varieties. We shall first consider the following situation. Let Y_1 and Y_2 be two irreducible varieties embedded in $\mathbb{P}^n$, for some $n \in 2\mathbb{N} + 1$. Assume that $\dim Y_1 = \dim Y_2 = k = \frac{n-1}{2}$. The join $J(Y_1, Y_2)$ has necessarily a joining component S of dimension $n - 1$, as shown in Lemma 6. We will show that if a third irreducible variety Y of the same dimension is such that $S \subset \Delta(Y)$, then the three varieties lie in the same $(k+1)$-dimensional linear subspace. Then we generalize to equidimensional varieties.

3.2.1. Two Varieties of Equal Dimension in a Space Whose Dimension Is Odd.

Theorem 1. *Let n be an odd number. Consider two distinct irreducible closed varieties Y_1 and Y_2 in $\mathbb{P}^3$, each of dimension $k = \frac{n-1}{2}$. By Lemma 6, consider the joining component S of $J(Y_1, Y_2)$, having dimension $n - 1$. If there exists a third irreducible variety Y of dimension k, distinct from Y_1 and Y_2, such that $S \subset \Delta(Y)$, then the three varieties lie in the same $(k+1)$-dimensional linear space, equal to the union of the lines in S.*

Proof. Step 1. Let

$$W = \bigcup_{l \in S} l.$$

By Lemma 7, W has dimension strictly greater than k. Moreover, the same lemma shows that the dimension of the variety of lines in S passing through a generic point p in Y has dimension k.

Step 2. Let l_0 be a generic line in S. Let $q_i = l_0 \cap Y_i$ and $p_0 = l_0 \cap Y$. Since l_0 is generic, these points can be assumed to be regular and $p_0 \notin Y_1 \cup Y_2$.

Let $\sigma_{p_0} \subset \mathbb{G}(1, n)$ be the set of lines passing through p_0. In general, $X_{p_0} = \sigma_{p_0} \cap S$ has dimension equal to k.

Consider now the morphism $f\colon Y_1 \to \sigma_{p_0}$, $a \mapsto a \vee p_0$. It is clear that $X_{p_0} \subset f(Y_1)$. Since the general fiber of f is finite, $\dim X_{p_0} = \dim Y_1$, and $f(Y_1)$ is irreducible, we have even the following equality: $X_{p_0} = f(Y_1)$. Therefore, f can be regarded as a morphism from Y_1 to X_{p_0}: $f\colon Y_1 \to X_{p_0}$, $a \mapsto a \vee p$. Here again the expression of the differential of f at q_1 is simply given by $df_{p_1}\colon T_{q_1}(Y_1) \to T_{l_0}(X_{p_0})$, $a \mapsto a \vee p_0$. The line l_0 being generic, we shall assume that $\dim T_{l_0}(X_{p_0}) = \dim X_{p_0} = k$.

Consider now

$$H_0 = \bigcup_{l \in T_{l_0}(X_{p_0})} l.$$

This linear space has dimension $k+1$. The expression for df_{p_1} shows that $T_{q_1}(Y_1) \subset H_0$. Similarly we can deduce that $T_{q_2}(Y_2) \subset H_0$. Therefore, the following inequality holds: $\dim(T_{q_1}(Y_1) \cap T_{q_2}(Y_2)) \geq k-1$.

By the same reasoning, there exists a dense open subset U of Y_1 such that for each $q \in U$, we have $\dim(T_q(Y_1) \cap T_{q_2}(Y_2)) \geq k-1$.

Step 3. If Y_2 is a linear space of dimension k, then by Proposition 2, Y_1 is contained in a $(k+1)$-dimensional linear space containing Y_2. A similar conclusion can be done if Y_1 is a linear space.

Step 4. Assume now that neither Y_1 nor Y_2 is a linear space. Applying the reasoning as in step 2 to X_{q_1} and X_{q_2}, which are, respectively, the sets of lines in S passing through q_1 and q_2, we get the following facts:

(1) there exists an open subset U_1 of Y_1 such that for all $q \in U_1$, we have $\dim(T_q(Y_1) \cap T_{p_0}(Y)) \geq k-1$;

(2) there exists an open subset U_2 of Y_2 such that for all $q \in Y_2$, we have $\dim(T_q(Y_2) \cap T_{p_0}(Y)) \geq k-1$.

When $k = 1$ (this is the case for curves in $\mathbb{P}^3$), these inequalities just mean that the intersections are not empty. Then by Proposition 2, each Y_i lies in a $(k+1)$-dimensional linear space Q_i containing $T_{p_0}(Y)$. These two linear spaces Q_1 and Q_2 are identical, since they are both generated by a line of S, namely l_0, and $T_{p_0}(Y)$. Let Q denote this linear space.

Then W, being the union of the lines in S, is included in Q. Thus, Y is also included in Q. Then every line in Q intersects the three varieties Y_1, Y_2, and Y. Therefore, the Fano variety of Q is the unique joining component of $J(Y_1, Y_2)$. The union of these lines is exactly Q. $\square$

3.2.2. Generalized Trisecant Lemma for Equidimensional Varieties. Since the proof is still valid if some or all of the varieties Y_1, Y_2, and Y are identical, we get a generalization of the trisecant lemma. We shall use the following notation: for a variety X, $V_{1,3}(X)$ is the closure in $\mathbb{G}(1,n)$ of

$$\{l \in \mathbb{G}(1,n) \mid \exists p, q, r \in X,\ p \neq q,\ p \neq r,\ q \neq r,\ p, q, r \in l\}.$$

Theorem 2 (the first generalization of the trisecant lemma)**.** *Let Z be a possibly singular equidimensional variety (maybe reducible or not) of dimension n, other than a linear space, embedded into $\mathbb{P}^r$, where $r \geq n+1$. The variety of trisecant lines of Z, i.e., $V_{1,3}(Z)$, has dimension strictly less than $2n$, unless Z is included in an $(n+1)$-dimensional linear space and has degree at least 3, in which case $\dim V_{1,3}(Z) = 2n$.*

Proof. Two cases must be considered.

Case 1. If $r < 2n+1$, then we can embed $\mathbb{P}^r$ into $\mathbb{P}^{2n+1}$ by a projective equivalence, so that we are in the setting of Theorem 1. Then the assertion follows immediately.

Case 2. In the case where $r \geq 2n+1$, let us define $s = r - 2n - 1 \geq 0$. We shall prove the result by induction over s. If $s = 0$, it is the content of Theorem 1.

Now it remains to show that if the result holds for some s, then it also holds for $s+1$. Let p be a generic point in $\mathbb{P}^r$, where $r = 2n+1+s+1$, and let H be any hyperplane in $\mathbb{P}^r$, not passing through p. Let Z' be the projection of Z over H through p. We can canonically identify H with $\mathbb{P}^{2n+1+s}$. Since the projection is generic and $\dim Z < r-1$, the general fiber of the projection $\pi\colon Z \to H$ is empty. However, over $\pi(Z)$, the general fiber is finite and nonempty. Therefore, the dimension of $V_{1,3}(Z')$ is also $2n$. Then, by the induction assumption, Z' is included in a linear space $L' \subset H$ of dimension $n+1$.

Let L be the space generated by p and L'. Then $\dim L = n+2$ and $Z \subset L$. Since $n+2 < 2n+1$, for $n > 1$, we can use the first step of the proof to conclude. Note that for $n = 1$, the result can be easily deduced from the classical trisecant lemma. $\square$

This result can also be expressed in the following terms.

Corollary. *Let Z be a variety of dimension n. If the variety of trisecant lines $V_{1,3}(Z)$ has dimension $2n$, then Z can be embedded into $\mathbb{P}^{n+1}$.*

3.3. Nonequidimensional Case. In this section, we turn to a more general case. Our purpose is to generalize Theorem 2 to the case where the variety Z is not equidimensional. As we proceeded before, we shall first inquire what happens with two irreducible varieties of complementary dimension.

3.3.1. A Two Varieties Statement. Let Y_1 and Y_2 be two irreducible closed varieties embedded in $\mathbb{P}^n$. Let us assume that $\dim Y_1 = k$ and $\dim Y_2 = n-1-k$, where $\frac{n-1}{2} \leq k \leq n-2$. The varieties Y_1 and Y_2 are assumed to be distinct. Let Y be another irreducible variety of dimension at most k, distinct from Y_1 and Y_2. By Lemma 6, let S be the joining component of $J(Y_1, Y_2)$, whose dimension is $n-1$. Let W be the subvariety of $\mathbb{P}^n$ being the union of the lines in S. This setting is used throughout Sec. 3.3. Our purpose is to show that W has dimension $k+1$.

The dimension of Y is k.

Lemma 8. *Let Y_1, Y_2, and Y be varieties defined as just above. If $S \subset \Delta(Y)$, then the dimension of Y must be equal to k.*

Proof. It is clear by Lemma 7. $\square$

We are now in a position to turn to the determination of the dimension of W.

W has dimension $k+1$.

Lemma 9. *Let Y_1, Y_2, and Y be varieties as in Lemma 8. Let q_1 and q_2 be generic points on Y_1 and Y_2, respectively. Let $p(q_1, q_2) = \overline{q_1 q_2} \cap Y$ be an intersection point of the line $\overline{q_1 q_2}$ and the variety Y. The points q_1, q_2, and $p(q_1, q_2)$ can be assumed to be regular. Then the tangent spaces $T_{q_1}(Y_1)$, $T_{q_2}(Y_2)$, and $T_{p(q_1,q_2)}(Y)$ lie in the same $(k+1)$-dimensional linear space.*

Proof. Step 1. The points q_1, q_2, and $p(q_1, q_2)$ can, indeed, be assumed to be regular, since the set of singular points of an algebraic variety is a proper closed subvariety [10].

First, let us prove that the line $\overline{q_1 q_2}$ and the tangent spaces $T_{q_1}(Y)$ and $T_{p(q_1,q_2)}(Y)$ lie in the same $(k+1)$-dimensional linear space.

Let $\sigma_{q_2} \subset \mathbb{G}(1, n)$ be the set of lines passing through q_2. In general, X_{q_2} has dimension equal to k (by Lemma 7).

Consider now the morphism $f\colon Y_1 \to \sigma_{q_2}$, $a \mapsto a \vee q_2$. For each $a \in Y_1$, the line $a \vee q_2$ lies in S. Therefore, f can be regarded as a morphism from Y_1 to X_{q_2}: $f\colon Y_1 \to X_{q_2}$, $a \mapsto a \vee q_2$. Again the differential of f at q_1 is given as follows: $df_{q_1}\colon T_{q_1}(Y_1) \to T_{\overline{q_1 q_2}}(X_{q_2})$, $a \mapsto a \vee q_2$.

Consider now

$$H_{q_1,q_2} = \bigcup_{l \in T_{\overline{q_1 q_2}}(X_{q_2})} l.$$

This linear space has dimension $k+1$. The expression of df_{q_1} shows that $T_{q_1}(Y_1) \subset H_{q_1,q_2}$. Thus, H_{q_1,q_2} is the $(k+1)$-dimensional linear space generated by $T_{q_1}(Y_1)$ and the line $\overline{q_1 q_2}$:

$$H_{q_1,q_2} = \langle T_{q_1}(Y_1), \overline{q_1 q_2} \rangle,$$

where $\langle\ \rangle$ denotes the linear span as in Terracini's lemma. Similarly, one can prove that $T_{p(q_1,q_2)}(Y) \subset H_{q_1,q_2}$.

Step 2. Consider now $\sigma_{p(q_1,q_2)}$, simply denoted σ_p below, the set of lines passing through $p(q_1, q_2)$.

Let $X_p = \sigma_p \cap S$. Lemma 7 shows that $\dim X_p = n-k-1$. Let $g\colon Y_2 \to \sigma_p$ be the morphism that sends a point $a \in Y_2$ to the line $a \vee p$, where $p = p(q_1, q_2)$. Since $X_p \subset g(Y_2)$, the general fiber of g is finite, $g(Y_2)$ is irreducible, and $\dim Y_2 = \dim X_p$, we see that the image of g is simply X_p. Thus, we can consider the morphism $g\colon Y_2 \to X_p$, $a \mapsto a \vee p$. The differential of g at q_2 gives rise to the morphism $dg_{q_2}\colon T_{q_2}(Y_2) \to T_{\overline{q_1 q_2}}(X_p)$ given by $a \mapsto a \vee p$.

Let

$$K_{q_1,q_2} = \bigcup_{l \in T_{\overline{q_1 q_2}}(X_p)} l$$

be the union of lines in $T_{\overline{q_1q_2}}(X_p)$. It has dimension $n-k$. The expression for dg_{q_2} shows that $T_{q_2}(Y) \subset K_{q_1,q_2}$.

Now let Z_1 be the subvariety of Y_1 defined as follows: $Z_1 = \{q \in Y_1 \mid \overline{qp} \in S\}$. It can be viewed as the trace on Y_1 of X_p. Let h be the morphism $h\colon Z_1 \to X_p$, $a \mapsto a \vee p$. Computing the differential of h at q_1, we see that $T_{q_1}(Z_1) \subset K_{q_1,q_2}$.

In view of $\dim T_{q_1}(Z_1) \geq n-k-1$, and, in general, $\overline{q_1q_2} \not\subset T_{q_1}(Z_1)$, and $\dim K_{q_1,q_2} = n-k$, we have $K_{q_1,q_2} = \langle T_{q_1}(Z_1), \overline{q_1q_2}\rangle$. Since $T_{q_1}(Z_1) \subset T_{q_1}(Y_1)$, we have $K_{q_1,q_2} \subset H_{q_1,q_2}$, whence $T_{q_2}(Y_2) \subset H_{q_1,q_2}$.

Thus, $T_{q_1}(Y_1)$, $T_{q_2}(Y_2)$, and $T_{p(q_1,q_2)}(Y)$, indeed, linearly span a $(k+1)$-dimensional linear space. $\square$

It is now possible to conclude using Terracini's lemma.

Theorem 3. *Let Y_1, Y_2, and Y be varieties as in Lemma* 8. *Then W must have dimension $k+1$.*

Proof. Consider smooth points $q_1 \in Y_1$ and $q_2 \in Y_2$. According to Lemma 9, the tangent spaces $T_{q_1}(Y_1)$ and $T_{q_2}(Y_2)$ linearly span, together with the line $\overline{q_1q_2}$, a $(k+1)$-dimensional linear space, which we shall denote K_{q_1,q_2}.

According to Terracini's lemma (Lemma 4), the tangent space of W at $\alpha q_1 + q_2$ for some $\alpha \neq 0$ lies in K_{q_1,q_2}. Thus, $\dim W \leq k+1$. Lemma 7 implies that $\dim W > k$. Therefore, we have $\dim W = k+1$. $\square$

In particular, the theorem shows that if W covers all the space, then there is no variety Y distinct from Y_1 and Y_2 that intersects every line in S.

Example.

We shall now proceed to show how one can construct varieties as in Sec. 3.3. For any k such that $\frac{n-1}{2} < k \leq n-2$, we can build varieties Y_1, Y_2, and Y satisfying the following conditions:

(1) $\dim Y_1 = \dim Y = k$;
(2) $\dim Y_2 = n-1-k$;
(3) $J(Y_1, Y_2)$ has a joining component S of dimension $n-1$;
(4) $S \subset \Delta(Y)$.

For this purpose, let $d = k-(n-1-k) = 2k-n+1 > 0$. Let $m > d$ be a natural number. Let Z_1 be a d-dimensional irreducible variety in $\mathbb{A}^m$, not passing through the origin. Let Z_2 be the single point variety made of the origin of $\mathbb{A}^m$. Let $f\colon \mathbb{A}^m \to \mathbb{A}^m$, $(a_1,\dots,a_m) \mapsto (a_1/2,\dots,a_m/2)$. Let $Z = f(Z_1)$. Consider now $\hat{Y}_1 = Z_1 \times \mathbb{A}^s$, $\hat{Y}_2 = Z_2 \times \mathbb{A}^s$, and $\hat{Y} = Z \times \mathbb{A}^s$.

If we take $s = k-d = n-k-1$ and $m = n-s = k+1 > d$, then we have the following conditions: $\dim \hat{Y}_1 = \dim \hat{Y} = k$, $\dim \hat{Y}_2 = n-k-1$, and $\hat{Y}_1, \hat{Y}_2, \hat{Y} \subset \mathbb{A}^n$.

Now we define Y_1, Y_2, and Y to be the projective closures of $\hat{Y}_1$, $\hat{Y}_2$, and $\hat{Y}$. Then by Lemma 6, we know that $J(Y_1, Y_2)$ has a joining component S of dimension $n-1$. Moreover, by construction we have $S \subset \Delta(Y)$ and $W = \bigcup_{l \in S} l$ has dimension $k+1$.

3.3.2. A General Statement. The proof being true even when $Y_2 \subset Y_1$ and $Y_1 = Y$, we get the following consequence, which can be regarded as a generalization of the trisecant lemma as well.

Theorem 4 (the second generalization of the trisecant lemma)**.** *Let Z be a possibly singular variety of dimension n, that may be neither irreducible nor equidimensional, embedded into $\mathbb{P}^r$, where $r \geq n+1$. Let Y be a proper subvariety of Z of dimension $k \geq 1$. Let S be an irreducible component of maximal dimension of $V_{1,3}(Y,Z)$, where $V_{1,3}(Y,Z)$ is the closure of*

$$\{l \in \mathbb{G}(1,r) \mid \exists p \in Y,\ q_1, q_2 \in Z \setminus Y,\ q_1 \neq q_2,\ p, q_1, q_2 \in l\}.$$

Then S has dimension strictly less than $n+k$ unless the union of lines in S has dimension $n+1$, in which case S has dimension $n+k$.

Proof. Step 1. The dimension of S is at most $n+k$, since $n+k$ is exactly the dimension of the join $J(Y, Z)$.

Step 2. If $r < n+k+1$, then we can embed $\mathbb{P}^r$ into $\mathbb{P}^{n+k+1}$ by a projective equivalence. According to Theorem 3, if $\dim S = n+k$, then the union of lines in S has dimension $n+1$.

Step 3. If $r \geq n+k+1$, then let $s = r-(n+k+1)$. If $s = 0$, the result holds by Theorem 3. Assume now that the result is true for some $s \in \mathbb{N}$; let us prove it for $s+1$.

The dimension r of the space can be expressed as $r = s+1+n+k+1$. Let p be a generic point in $\mathbb{P}^{s+1+n+k+1}$ and H be a hyperplane not passing through p. Then let Z' (Y') be the projection of Z (respectively, Y) over H through p. Then Z' is embedded into a projective space of dimension $s+n+k+1$. The general fiber of the projection $\pi\colon Z \to Z'$ is finite.

Each line in S is projected onto a line of the closure $V_{1,3}(Y', Z')$ of

$$\{l \in \mathbb{G}(1, r-1) \mid \exists p \in Y',\ q_1, q_2 \in Z' \setminus Y',\ q_1 \neq q_2,\ p, q_1, q_2 \in l\}.$$

Let $S' \subset V_{1,3}(Y', Z')$ be defined as consisting of those lines that are built by the projection of lines in S. Since the general fiber of π is finite, we see that $\dim(S') = \dim(S)$.

Therefore, if $\dim S = n+k$, then $\dim S' = n+k$. In that case, since $\dim J(Y', Z') = n+k$, S' must be an irreducible component of maximal dimension of $V_{1,3}(Y'Z') \subset J(Y', Z')$. Thus, by the induction assumption, $W' = \bigcup\limits_{l \in S'} l$ has dimension $n+1$ and so $\dim W = n+1$, because the general fiber of $\pi\colon W \to W'$ is finite. $\square$

Note that if $r > n+1$ and $\dim(S) = n+k$, then the theorem implies that the union of lines in S cannot cover the whole space.

Example.

We shall now conclude by giving an example of an n-dimensional variety with k-secant lines variety of dimension $2n-1$, for $k \geq 3$. This improves the well-known construction, also presented in [9], of n-dimensional varieties admitting an $(n+1)$-dimensional variety of k-secant lines.

Let $p \in \mathbb{A}^3$ be the origin and consider an irreducible curve $X_1 \subset \mathbb{A}^3$ not passing through p. For $m \in \mathbb{N}$, where $m \geq 2$, let X_m be $f_m(X_1)$, where $f_m(x, y, z) = (mx, my, mz)$. For each $m \geq 1$, we denote $Y_m = X_m \times \mathbb{A}^{n-1}$. For a given $k \geq 3$, we define $Z_k = \bigcup\limits_{1 \leq m \leq k} Y_m$. Then $\dim Z_k = n$ and Z_k admits a family of k-secant lines whose dimension is $2n-1$.

We can also find an irreducible variety Z containing Z_k and having dimension $n' = n+1$. For this variety, the family of lines has dimension $2n'-3$.

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