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Automorphisms of the endomorphism semigroup of a polynomial algebra

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ABSTRACT

We describe the automorphism group of the endomorphism semigroup $\mathrm{End}(K[x_1,\dots,x_n])$ of ring $K[x_1,\dots,x_n]$ of polynomials over an *arbitrary* field K. A similar result is obtained for automorphism group of the category of finitely generated free commutative–associative algebras of the variety $\mathcal{CA}$ commutative algebras. This solves two problems posed by B. Plotkin (2003) [18, Problems 12 and 15].

More precisely, we prove that if $\varphi \in \mathrm{Aut}\,\mathrm{End}(K[x_1,\dots,x_n])$ then there exists a semi-linear automorphism $s : K[x_1,\dots,x_n] \to K[x_1,\dots,x_n]$ such that $\varphi(g) = s \circ g \circ s^{-1}$ for any $g \in \mathrm{End}(K[x_1,\dots,x_n])$. This extends the result obtained by A. Berzins for an infinite field K.

1. Introduction

We describe the group $G = \mathrm{Aut}\,\mathrm{End}(K[x_1,\dots,x_n])$, where K is an arbitrary field. A similar result is obtained also for automorphism group of the category of finitely generated free commutative–associative algebras of the variety commutative algebras. This solves two problems posed by B. Plotkin [18, Problems 12 and 15].

More precisely, we prove that if $\varphi \in \mathrm{Aut}\,\mathrm{End}(K[x_1,\dots,x_n])$ then there exists a semi-linear automorphism $s : K[x_1,\dots,x_n] \to K[x_1,\dots,x_n]$ such that $\varphi(g) = s \circ g \circ s^{-1}$ for any $g \in \mathrm{End}(K[x_1,\dots,x_n])$ (see Theorem 3.6). Here "semi-linearity" means that s is a composition of an automorphism of the

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field K and an automorphism of the ring $K[x_1,\dots,x_n]$. We note that for an infinite ground field K such result was obtained earlier by A. Berzins [3].

A problem of description of the group $G = \mathrm{Aut}\,\mathrm{End}(K[x_1,\dots,x_n])$ is also interesting in the context of Universal Algebraic Geometry (UAG). Let Θ be a variety of algebras over a field K and $F = F(X)$ be a free algebra from Θ generated by a finite subset X of some infinite universum X^0. We refer to [17,18] (see also [8]) for the Universal Algebraic Geometry (UAG) notions used in our work.

If an algebra G belongs to Θ one can consider the category of algebraic sets $K_\Theta(G)$ over G. Objects of this category are algebraic sets in affine space over G; the category $K_\Theta(G)$ defines a geometry of the algebra G in Θ. One of the main problems in UAG is to determine whether two different algebras G_1 and G_2 have the same geometry. The coincidence of geometries means that the categories $K_\Theta(G_1)$ and $K_\Theta(G_2)$ are equivalent. It is known that coincidence of geometries of G_1 and G_2 is determined by the structure of the group $\mathrm{Aut}\,\Theta^0$, where Θ^0 is the category of free finitely generated algebras of Θ. On the other hand, there is a natural relation between the structure of the groups $\mathrm{Aut}\,\mathrm{End}\,F$ and $\mathrm{Aut}\,\Theta^0$. The structure of the latter is determined by the group $\mathrm{Aut}\,\mathrm{End}\,F$. It should be mentioned that a problem of investigation of the groups $\mathrm{Aut}\,\mathrm{End}\,F$, $F \in \Theta$, for different varieties Θ is quite interesting by itself and has been considered in many papers (see [1–3,5,8–11,13–19,23]).

Let $\mathcal{CA}$ be the variety of a commutative–associative algebras with 1 over a field K, $A = K[x_1,\dots,x_n]$ be a free commutative–associative algebra in $\mathcal{CA}$ freely generated over K by a set $X = \{x_1,\dots,x_n\}$, i.e., a polynomial algebra in variables $x_1,\dots,x_n$. In this work we obtain a description of the group $\mathrm{Aut}\,\mathcal{CA}^0$ of automorphisms of the category $\mathcal{CA}^0$. Note that this description is a generalization of previous result on the structure of $\mathrm{Aut}\,\mathcal{CA}^0$ for the variety $\mathcal{CA}$ of a commutative–associative algebras over an infinite field K [3].

Our description is based on new characteristics of endomorphisms of A such as *rank* of endomorphisms of A. We discuss external and internal definitions of this notation. The former is expressed in terms of the action of the semigroup $\mathrm{End}\,A$ on A, while the latter can be written in terms of the semigroup itself. This approach allows us to describe the above mentioned properties of endomorphisms of A in an invariant manner and paves the way for proof of the main assertions in the paper: the group $\mathrm{Aut}\,\mathrm{End}\,A$ is generated by semi-inner automorphisms of $\mathrm{End}\,A$.

Our approach employs this technique (developed in [5,9]) supplemented by algebro-geometric methods of investigations.

2. On the endomorphism semigroup of a free associative–commutative algebra

2.1. Rank of an endomorphism of polynomial algebra

Let $A = K[x_1,\dots,x_n]$ be a free commutative–associative algebra over a field K generated by $X = \{x_1,\dots,x_n\}$ (below *polynomial algebra* over K in variables X). Earlier, in [5], we defined *the endomorphism* of free associative algebra $K\langle x_1,\dots,x_n\rangle$ *of rank* 0 *and* 1. In this section we introduce a definition of *endomorphisms of arbitrary rank m* in a polynomial algebra $K[x_1,\dots,x_n]$.

First, we introduce the “external” and “internal” definitions of *rank* of endomorphism φ of algebra A and show their equivalence.

Definition 2.1 *(“External” definition of an endomorphism of rank m).* An endomorphism

$$\varphi : A \to A$$

has **rank** m if $\mathrm{trdeg}(\mathrm{Im}\,\varphi) = m$, i.e., the transcendence degree of the K-algebra $M = \mathrm{Im}\,\varphi \subseteq A$ is equal to m. We denote this as $\mathrm{rk}(\varphi) = m$. It is evident that there exist endomorphisms of $K[x_1,\dots,x_n]$ of arbitrary rank $\leqslant n$. For instance, the identical mapping on $K[x_1,\dots,x_n]$ is the endomorphism of rank n.

For the internal definition of rank m endomorphisms, we need to define a congruence on the semigroup $\mathrm{End}(A)$ with respect to a fixed endomorphism φ of A.

Definition 2.2. Endomorphisms φ_1 and φ_2 of A are *φ-equivalent* if $\varphi\varphi_1 = \varphi\varphi_2$. In this case we write $\varphi_1 \backsim_\varphi \varphi_2$.

It is clear that $\backsim_\varphi$ is an equivalence relation on $\operatorname{End} A$. Let S be the set of all φ-equivalences on $\operatorname{End} A$. We determine the preorder $\trianglelefteq$ on the set S as follows. We say that $\backsim_\phi \trianglelefteq \backsim_\psi$, where $\phi, \psi \in \operatorname{End} A$, if

$$\phi\varphi_1 = \phi\varphi_2 \quad \Rightarrow \quad \psi\varphi_1 = \psi\varphi_2,$$

for any $\varphi_1, \varphi_2 \in \operatorname{End} A$. The preorder $\trianglelefteq$ can be extended up to the order $\preccurlyeq$ on the quotient set $\widetilde{S} = S/R$ under equivalence R, where $\backsim_\phi R \backsim_\psi$ if and only if $\backsim_\phi \trianglelefteq \backsim_\psi$ and $\backsim_\psi \trianglelefteq \backsim_\phi$. Denote by $\backsim_{\psi_R}$ the R-equivalence class of a relation $\backsim_\psi$.

Definition 2.3. We say that $\phi \preccurlyeq \psi$ iff $\backsim_{\phi_R} \preccurlyeq \backsim_{\psi_R}$.

Definition 2.4. We say that $\phi \prec \psi$ if $\backsim_{\phi_R} \preccurlyeq \backsim_{\psi_R}$ and $\backsim_{\psi_R} \nsim \backsim_{\phi_R}$.

It is clear that relations $\preccurlyeq$ and $\prec$ are an order and a strong order, respectively, on $\operatorname{End} A$. Note that the smaller endomorphism φ (in the sense of $\preccurlyeq$) corresponds to the stronger equivalence relation $\sim_\varphi$. The proof of the following lemma is straightforward.

Lemma 2.5. *Let $\varphi = (\varphi_1(\vec{x}), \dots, \varphi_n(\vec{x}))$ and $\phi = (\psi_1(\vec{x}), \dots, \psi_n(\vec{x}))$ be two endomorphisms of $K[x_1, \dots, x_n]$. Then*

(1) *$\phi \sim \psi$ iff for all $H(\vec{x}) \in K[x_1, \dots, x_n]$ the condition $H(\varphi_1(\vec{x}), \dots, \varphi_n(\vec{x})) = 0$ is equivalent to $H(\psi_1(\vec{x}), \dots, \psi_n(\vec{x})) = 0$.*
(2) *$\phi \preccurlyeq \psi$ iff for all $H(\vec{x}) \in K[x_1, \dots, x_n]$ the condition $H(\varphi_1(\vec{x}), \dots, \varphi_n(\vec{x})) = 0$ implies $H(\psi_1(\vec{x}), \dots, \psi_n(\vec{x})) = 0$.*
(3) *$\phi \prec \psi$ iff for all $H(\vec{x}) \in K[x_1, \dots, x_n]$ the condition $H(\varphi_1(\vec{x}), \dots, \varphi_n(\vec{x})) = 0$ implies $H(\psi_1(\vec{x}), \dots, \psi_n(\vec{x})) = 0$ and there exists $R(\vec{x}) \in K[x_1, \dots, x_n]$ such that $R(\varphi_1(\vec{x}), \dots, \varphi_n(\vec{x})) = 0$ but $H(\psi_1(\vec{x}), \dots, \psi_n(\vec{x})) \neq 0$.*

Definition 2.6 *("Internal" definition of an endomorphism of rank m).* An endomorphism $\psi : A \to A$ is of *rank m*, if maximum of the lengths of all chains of endomorphisms of A of the form

$$\psi \precnsim \psi_{m-1} \precnsim \cdots \precnsim \psi_1 \precnsim \psi_0, \tag{2.1}$$

is equal to m. If there is no endomorphism ψ such that $\psi \precnsim \psi_0$, then ψ has *rank* 0.

Remark 2.7. If $\operatorname{rk}(\varphi) = 0$, then image of φ is the ground field. The definition of endomorphisms of rank 0 and 1 for associative–commutative algebra is in accordance with the definition for a free associative algebra given in [5]. The internal definition of rank 0 is pretty similar.

Proposition 2.8. *Definitions* 2.6 *and* 2.1 *are equivalent.*

We precede the proof of this proposition by several lemmas. Denote by $\mathbf{A}_K^n$ an n-dimensional affine space over the algebraic closure $\bar{K}$ of the field K. It is clear that $\mathbf{A}_K^n \simeq \operatorname{Specm}(K[x_1, \dots, x_n])$, where $\operatorname{Specm}(K[x_1, \dots, x_n])$ is the set of all maximal ideals of $K[x_1, \dots, x_n]$. Let us investigate the algebro-geometric properties of polynomial endomorphisms of $K[x_1, \dots, x_n]$ and their relation to polynomial maps of $\mathbf{A}_K^n$ into itself.

Each endomorphism $\varphi : K[x_1,\dots,x_n] \to K[x_1,\dots,x_n]$ such that

$$\varphi(x_i) = \varphi_i(x_1,\dots,x_n), \quad \text{where } \varphi_i = \varphi_i(x_1,\dots,x_n) \in K[x_1,\dots,x_n],$$

determines a polynomial map $\varphi^* = (\varphi_1,\dots,\varphi_n) : \mathbf{A}_K^n \to \mathbf{A}_K^n$ of the affine space $\mathbf{A}_K^n$ into itself of the form

$$(x_1,\dots,x_n) \to \bigl(\varphi_1(x_1,\dots,x_n),\dots,\varphi_n(x_1,\dots,x_n)\bigr). \tag{2.2}$$

The converse is also true: to each polynomial map $\varphi^* : \mathbf{A}_K^n \to \mathbf{A}_K^n$ of the form (2.2) corresponds to the above mentioned endomorphism φ of the algebra $K[x_1,\dots,x_n]$. We will make use of this relation below.

Denote by M_φ the variety $\varphi^*(\mathbf{A}_K^n)$. We shall say that the variety M_φ *corresponds* to the endomorphism φ of the polynomial algebra $K[x_1,\dots,x_n]$. The coordinate ring $K[M_\varphi]$ of the variety M_φ is $K[M_\varphi] = K[x_1,\dots,x_n]/I$, where

$$I = \bigl\{H(x_1,\dots,x_n) \bigm| H\bigl(\varphi_1(\vec{x}),\dots,\varphi_n(\vec{x})\bigr) = 0\bigr\}$$

is the ideal in $K[x_1,\dots,x_n]$ corresponding to the variety M_φ. It is clear that $K[M_\varphi] \simeq K[\varphi_1(\vec{x}),\dots,\varphi_n(\vec{x})]$ and $\dim M_\varphi = \operatorname{trdeg} K[\varphi_1(\vec{x}),\dots,\varphi_n(\vec{x})]$.

Lemma 2.9. *The variety M_φ is irreducible.*

Proof. Since the affine variety $\mathbf{A}_K^n$ corresponding to the algebra $K[x_1,\dots,x_n]$ is irreducible and the image of an irreducible algebraic variety is also irreducible [6,22], the variety M_φ is irreducible. Hint: coordinate ring of an image isomorphic to subring of the coordinate ring of the preimage, hence has no zero divisors. □

Lemma 2.10. *Let ϕ_1, ϕ_2 be endomorphisms of $K[x_1,\dots,x_n]$ and M_{ϕ_1}, M_{ϕ_2} be two corresponding varieties, respectively. The following properties hold*:

(1) *If $\phi_1 \sim \phi_2$, then $M_{\phi_1} \cong M_{\phi_2}$ and the corresponding coordinate rings are isomorphic.*
(2) *$\phi_1 \preccurlyeq \phi_2$ if and only if the coordinate ring of M_{ϕ_1} is a quotient ring of the coordinate ring of M_{ϕ_2}. In this case* $\dim M_{\phi_2} \leqslant \dim M_{\phi_1}$, *where* $\dim X$ *is the Krull dimension of a variety X. If the quotient ring is proper, then the inequality is strict.*

Proof. (1) By item (3) of Lemma 2.5, the coordinate rings of the varieties M_{ϕ_1} and M_{ϕ_2} are isomorphic. Therefore, the above varieties themselves are isomorphic.

(2) By item (2) of Lemma 2.5, the coordinate ring of the variety M_{ϕ_1} is a quotient ring of the coordinate ring of the variety M_{ϕ_2} by some its ideal. As a consequence, $\dim M_{\phi_1} \leqslant \dim M_{\phi_2}$ (see also [6,22]). □

Let ψ be an endomorphism of $K[x_1,\dots,x_n]$ of "external" rank m. The last lemma shows that there exist no chains of endomorphisms ψ_i of the form (2.1) of length more than m beginning with ψ. It means that the inner rank of ψ is less or equal than the outer its rank. In order to prove Proposition 2.8 we need to establish an opposite inequality, i.e., to prove that there exists a chain (2.1) of length m beginning with ψ.

Lemma 2.11. *Notations being as above, let* $\dim M_\varphi = m$. *Then there exists an endomorphism φ' of $K[x_1,\dots,x_n]$ such that $\varphi' \prec \varphi$ and $\dim M_{\varphi'} = m - 1$.*

The assertion of this lemma is evident for $m = 1$: in this case it is sufficient to consider specialization $x_i \to \xi_i$, $\xi_i \in K$, into ground field K.

Now we pass to the general case. We need the following lemma:

Lemma 2.12. *Let R be a subalgebra of $K[x_1, \dots, x_n]$ of a transcendence degree m ($m \leqslant n$). Then there exists an embedding from R into $K[x_1, \dots, x_m]$.*

Remark 2.13. A similar statement for field embeddings was established in [4].

Proof of Lemma 2.12. It is known that any transcendence base of a subalgebra A of an algebra B can be extended to a transcendence base of the algebra B. Let $y_1, \dots, y_m$ be a transcendence base of R. We can complete this base to a base $y_1, \dots, y_m, z_1, \dots, z_{n-m}$ of $K[x_1, \dots, x_n]$. It is clear that the elements $z_1, \dots, z_{n-m}$ are algebraically independent over R and they generate a subalgebra $R[z_1, \dots, z_{n-m}]$ of $K[x_1, \dots, x_n]$. Therefore, the affine domain $R[z_1, \dots, z_{n-m}]$ can be embedded into an affine domain $K[x_1, \dots, x_m][x_1, \dots, x_{n-m}]$. However, it is known that if A and B are two domains such that $A[x_1, \dots, x_s]$ can be embedded into $B[x_1, \dots, x_s]$, then A can be embedded into B (see [4]). Therefore, R can be embedded into the polynomial algebra $K[x_1, \dots, x_m]$. □

Now, by Lemma 2.12 one can assume that polynomials $\varphi_1, \dots, \varphi_n$ defining the mapping φ belong to $K[x_1, \dots, x_m]$ and $\operatorname{trdeg}(\varphi_1, \dots, \varphi_n) = m$, $m \leqslant n$.

Lemma 2.14. *Let $\varphi_1(x_1, \dots, x_m), \dots, \varphi_n(x_1, \dots, x_m)$, where $n \geqslant m$, be a collection of polynomials from $K[x_1, \dots, x_m]$ which generates the subalgebra of $K[x_1, \dots, x_n]$ of transcendence degree m. Then for any specialization $x_m \to \xi$, $\xi \in K$, except a finite set of values of $\xi \in K$, the algebra $K[\varphi_1(x_1, \dots, x_{m-1}, \xi), \dots, \varphi_n(x_1, \dots, x_{m-1}, \xi)]$ has the transcendence degree $m - 1$.*

Proof. Without loss of generality it is sufficient to consider the case when K is an algebraically closed field (tensoring over algebraic closure, if necessary). Consider a mapping $\Phi : \mathbf{A}_K^m \to \mathbf{A}_K^{n+1}$ such that $\Phi(\vec{x}) = (\varphi_1(\vec{x}), \dots, \varphi_n(\vec{x}), x_m)$ where $\vec{x} = (x_1, \dots, x_m)$. Denote by M the image of Φ. Since $\operatorname{trdeg}(\varphi_1, \dots, \varphi_n) = m$ and the dimension of image Φ is at most m, we have $\dim M = m$. Now we consider a projection $\pi : \mathbf{A}_K^{n+1} \to \mathbf{A}_K^1$ such that $\pi(z_1, \dots, z_n, x_m) = x_m$. Denote by π_1 the restriction of π to M. It is clear that π_1 is an epimorphic mapping. Further we use the following

Theorem 2.15. *(See [6,22].) If $f : X \to Y$ is a regular mapping between irreducible varieties X and Y: $f(X) = Y$, $\dim X = n$, $\dim Y = m$, then $m \leqslant n$ and*

(1) *$\dim f^{-1}(y) \geqslant n - m$ for every point $y \in Y$.*
(2) *There exists a non-empty set $U \subset Y$ such that $\dim f^{-1}(y) = n - m$ for all $y \in U$.*

In our case $Y = \mathbf{A}_K^1$, $\dim Y = 1$, $\dim X = m$. Therefore, for all points of $\mathbf{A}_K^1$, except points of closed subvariety T of $\mathbf{A}_K^1$, the fiber $\pi^{-1}(\xi)$ has the dimension $m - 1$. Therefore,

$$\operatorname{trdeg} K\bigl[P_1(x_1, \dots, x_{m-1}, \xi), \dots, P_n(x_1, \dots, x_{m-1}, \xi)\bigr] = m - 1,$$

except a finite set of $\xi \in K$. This concludes the proof of Lemma 2.14. □

Remark 2.16. A proof of Lemma 2.11 follows immediately from the above lemma in the **case of an infinite ground field**. Indeed, if a field K is infinite, by Lemma 2.14 we can choose $\xi \in K$ such that $\varphi_1' = \varphi_1(x_1, x_2, \dots, x_{n-1}, \xi), \dots, \varphi_n' = \varphi_n(x_1, \dots, x_{n-1}, \xi)$ and $\operatorname{trdeg} K[\varphi_1'(\vec{x}), \dots, \varphi_n'(\vec{x})] = m - 1$. As a corollary, we have $\dim M_{\varphi'} = k - 1$, where $\varphi' = (\varphi_1', \dots, \varphi_n')$. Hence, our Lemma 2.11 is proven in the case of an infinite field. This provides a description of the group $\operatorname{Aut}(\operatorname{End}(K[x_1, \dots, x_n]))$ for the case of an infinite ground field K as was obtained earlier by Berzins [3].

However, in the case of a finite ground field there can be no such small jumps from φ_i to φ_i', such that $\dim M_{\varphi'} = \dim M_\varphi - 1$, for any specialization of variables into a ground field K.

Example 2.17. Let $|K| = q$ and $\varphi_i = \prod_{k=1}^n (x_k^q - x_k) \cdot x_i$. It is evident that $\operatorname{trdeg}(\varphi_1, \dots, \varphi_n) = n$. However, any specialization of φ_i of the form: $x_n \to \xi$, $\xi \in K$, yields us $\varphi_i' = 0$.

If a field K is finite *instead of specializations of x_n into ground field we consider substitutions into polynomials depending on other variables, in particular, on powers of other variables.* We need the following

Theorem 2.18. *(See [4].) Letting $\xi_1, \dots, \xi_s$ be algebraic over $K[x_1, \dots, x_m]$, the polynomials $Q_i(\vec{t}, \vec{x}, \vec{\xi})$, $i = 1, \dots, n$, are algebraically independent for some value of set of parameter $\vec{t} = (t_1, \dots, t_n)$ in some extension field k_1 of the ground field k. Then there exist polynomials $R_i \in \Phi[x_1]$, $i = 1, 2, \dots, r$, $\vec{R} = (R_1, \dots, R_r)$ such that the set of polynomial*

$$\{Q_1(\vec{t}, \vec{x}, \vec{\xi}), \dots, Q_n(\vec{t}, \vec{x}, \vec{\xi})\}$$

is algebraically independent. Moreover, if the growth of the sequence

$$n_1 \ll n_2 \ll \cdots \ll n_r$$

is sufficiently large, we may assume $R_i = x_1^{n_i}$. The above statement is still valid if we replace "$k[x_1, \dots, x_m]$" by "$k(x_1, \dots, x_m)$" and "polynomial" for rational function. In this case we can put $R_i = x_1^{-n_i}$.

Instead of x_1 one can take any other variable x_i; $\Phi = \mathbb{Z}_p$ if $\operatorname{char} K = p$ and $\Phi = \mathbb{Z}$ if $\operatorname{char} K = 0$.

We use a special case of this theorem for $r = 1$ and $s = 0$, i.e., a variant of this theorem without ξ_i. The next assertion is also needed for the proof of Lemma 2.11 in the case of a finite ground field K.

Assertion 2.19. *Let $Q_1(x_1, \dots, x_m), \dots, Q_n(x_1, \dots, x_m)$ be a set of polynomials from $K[x_1, \dots, x_m]$, $|K| < \infty$, and the transcendence degree of the algebra*

$$K[Q_1(x_1, \dots, x_m), \dots, Q_n(x_1, \dots, x_m)]$$

equal to m, where $1 < m \leqslant n$. If $r \in \mathbb{N}$ is sufficiently large, then

$$\operatorname{trdeg}\big(K\big[Q_1\big(x_1, \dots, x_1^r\big), \dots, Q_n\big(x_1, \dots, x_1^r\big)\big]\big) = m - 1.$$

Proof. Denote $A = K[Q_1(x_1, \dots, x_{m-1}, x_1^r), \dots, Q_n(x_1, \dots, x_{m-1}, x_1^r)]$. It is clear that $A \subseteq K[x_1, \dots, x_{m-1}]$, i.e., $\operatorname{trdeg}(A) \leqslant m - 1$. We have to prove that the opposite inequality is also fulfilled for sufficiently large r. Since

$$\operatorname{trdeg}\big(K\big[Q_1(x_1, \dots, x_m), \dots, Q_n(x_1, \dots, x_m)\big]\big) = m,$$

we can choose m algebraically independent polynomials between Q_i. Without loss of generality, we can set that these polynomials are $Q_1, \dots, Q_m$. By Lemma 2.14, there exists $\eta \in \bar{K}$, where $\bar{K}$ is the algebraic closure of field K, such that

$$\operatorname{trdeg}\big(\bar{K}\big[Q_1(x_1, \dots, x_{m-1}, \eta), \dots, Q_m(x_1, \dots, x_{m-1}, \eta)\big]\big) = m - 1.$$

Without loss of generality, we can suppose that the first $m - 1$ polynomials $Q_i(x_1, \dots, x_{m-1}, \eta)$, $1 \leqslant i \leqslant m - 1$, are algebraically independent over $\bar{K}$. By Theorem 2.18, there exists a natural r_0, such that the polynomials

$$Q_1\left(x_1, \ldots, x_{m-1}, x^r\right), \quad \ldots, \quad Q_{m-1}\left(x_1, \ldots, x_{m-1}, x^r\right)$$

are algebraically independent over K for any $r \geqslant r_0$. Since the dimension of the subring $K[Q_1(x_1, \ldots, x_{m-1}, x^r), \ldots, Q_{m-1}(x_1, \ldots, x_{m-1}, x^r)]$ is not less than the dimension of its subring $K[Q_1(x_1, \ldots, x_{m-1}, x^r), \ldots, Q_n(x_1, \ldots, x_{m-1}, x^r)]$, the proof is complete. $\square$

We summarize our results in the following

Assertion 2.20. *Let $\varphi = (\varphi_1(x_1, \ldots, x_n), \ldots, \varphi_n(x_1, \ldots, x_n))$ be an endomorphisms of $K[x_1, \ldots, x_n]$ of "internal" rank m. Then there exists an endomorphism $\psi = (\psi_1(x_1, \ldots, x_m), \ldots, \psi_n(x_1, \ldots, x_m))$, $\psi_i(x_1, \ldots, x_m) \in K[x_1, \ldots, x_m]$, such that $\varphi \sim \psi$. In addition, an endomorphism*

$$\psi'_{(r)} = \left(\psi_1\left(x_1, \ldots, x_{m-1}, x_1^r\right), \ldots, \psi_n\left(x_1, \ldots, x_{m-1}, x_1^r\right)\right)$$

has the rank at most $m-1$ for any $r \in \mathbb{N}$. Moreover, there exists $r_0 \in \mathbb{N}$ such that for all $r \geqslant r_0$ holds: $\psi'_{(r)} \prec \psi$. As a consequence, $\psi'_{(r)} \prec \varphi$ and an "internal" rank of $\psi'_{(r)}$ is equal to $m-1$ for all $r \geqslant r_0$.

With these assertions, the proof of Lemma 2.11 is straightforward. Now we are ready to prove Proposition 2.8.

Proof of Proposition 2.8. Suppose that φ has an "internal" rank m, i.e., there exists a maximal chain of length m beginning with φ:

$$\varphi \precnsim \varphi_{m-1} \precnsim \cdots \precnsim \varphi_1 \prec \varphi_0. \tag{2.3}$$

We have a descending chain of the corresponding varieties M_{φ_i}:

$$M_{\varphi_0} \subseteq M_{\varphi_1} \subseteq \cdots \subseteq M_{\varphi_{m-1}} \subseteq M_{\varphi}. \tag{2.4}$$

The induction argument on the length m of the chain (2.4) leads us to the case $m = 0$ for which our assertion is evident. Therefore, the "external" rank of φ is also equal to m.

Conversely, let an endomorphism φ be of "external" rank m, i.e., $\operatorname{trdeg} \operatorname{Im} \varphi = m$. By Lemma 2.11, there exists an endomorphism ψ_{m-1} of $K[x_1, \ldots, x_n]$ such that $\psi_{m-1} \prec \varphi$ and $\dim M_{\psi_{m-1}} = m-1$. In the same way, we can construct a chain of the form (2.3) beginning with φ. It is clear that this chain has the length m, as desired. $\square$

Since the chain (2.1) is invariant under automorphisms of $\operatorname{End} K[x_1, \ldots, x_n]$, we have

Corollary 2.21. *Let $\Phi \in \operatorname{Aut}(\operatorname{End}(A))$, $\psi \in \operatorname{End}(A)$, and $\operatorname{rk}(\psi) = m$. Then $\operatorname{rk}(\Phi(\psi)) = m$.*

Remark 2.22. Below we need endomorphisms of rank 0 and 1. By Definition 2.1, an endomorphism ψ of A is of rank 0 if $\psi(A) = K$. An endomorphism φ of A is of rank 1 if $\operatorname{trdeg}(\operatorname{Im} \varphi) = 1$. It is known [4,21], that every integrally closed subalgebra B of $A = K[x_1, \ldots, x_n]$ of transcendence degree 1 is isomorphic to a polynomial algebra $K[t]$ in variable t. Taking into account that the integer closure B of the algebra $\varphi(A)$ in A is an algebra of the same transcendence degree as $\varphi(A)$, we conclude that the algebra B is isomorphic to a polynomial algebra $K[t]$ in variable t. As a consequence, the algebra $\varphi(A)$ is a polynomial algebra $K[y]$, where y is an element in $K[x_1, \ldots, x_n]$.

2.2. Representations of Kronecker semigroup of rank n

Recall the definition of Kronecker endomorphisms of the free associative algebra A.

Definition 2.23. (Cf. [9,11].) *Kronecker endomorphisms* of A in the base $X = \{x_1, \dots, x_n\}$, $x_i \in A$, are the endomorphisms e_{ij}, $i, j \in [1n]$, of A which are determined on free generators $x_k \in X$ by the rule: $e_{ij}(x_k) = \delta_{jk} x_i$, $x_i \in X$, $i, j, k \in [1n]$ and δ_{jk} is the Kronecker delta.

It is clear that any Kronecker endomorphism of A has rank 1.

Definition 2.24. A semigroup Γ_n with an adjoint zero element 0 generated by b_{ij}, $ij \in [1n]$, with defining relations

$$b_{ij} \cdot b_{km} = \delta_{jk} b_{im}, \qquad b_{ij} \cdot 0 = 0 \cdot b_{ij} = 0$$

is called a *Kronecker semigroup of rank n*.

Denote by E_n a semigroup generated by e_{ij}, $i, j \in [1n]$, and an adjoint zero. Clearly, the semigroup E_n is a Kronecker semigroup of rank n.

Remark 2.25. We have a notion of the rank of a Kronecker semigroup Γ. Don't confuse it with the rank of an endomorphism of A.

Definition 2.26. *A representation of a semigroup T* in the semigroup $\operatorname{End} A$ is a homomorphism $\nu : T \to \operatorname{End} A$.

Definition 2.27. Let $\rho : \Gamma_n \to \operatorname{End} A$ be a representation of the Kronecker semigroup Γ of rank n in $\operatorname{End} A$. We say that the representation ρ is *singular* if $\operatorname{rk} \rho(b_{ij}) = 0$ for any $i, j \in [1n]$.

In fact, it is sufficient to require that $\operatorname{rk} \rho(b_{11}) = 0$.

Proposition 2.28. *Let $\rho : \Gamma_n \to \operatorname{End} A$ be a singular representation of the Kronecker semigroup Γ of rank n in* $\operatorname{End} A$ *and $q = \rho \cdot \rho^{-1}$ the kernel congruence on Γ_n. Then $\Gamma_n/q \cong A$, where $A = \langle \varphi \rangle$ is a one-element semigroup such that $\rho(0) = \varphi$, $\varphi \in \operatorname{End} A$, and $\operatorname{rk}(\varphi) = 0$. Conversely, if $\varphi \in \operatorname{End} A$ is an endomorphism of rank* 0*, then there exists a representation $\rho : \Gamma_n \to \operatorname{End} A$ such that $\rho(0) = \varphi$.*

Proof. From $0 \cdot b_{ij} = 0$, $i, j \in [1n]$, it follows $\varphi\rho(b_{ij}) = \varphi$, where $\rho(0) = \varphi$. Since φ is the identical mapping on K and $\operatorname{rk}(\rho(b_{ij})) = 0$, we have $\rho(b_{ij}) = \varphi$ for any $i, j \in [1n]$. Thus, $\Gamma_n/q \cong A$, where $A = \langle \varphi \rangle$.

Conversely, if φ is an endomorphism of $\operatorname{End} A$ such that $\operatorname{rk}(\varphi) = 0$, define a representation $\rho : \Gamma_n \to \operatorname{End} A$ by the rule $\rho(0) = \rho(b_{ij}) = \varphi$ for all $i, j \in [1n]$. It is clear that we obtained a required representation ρ. $\square$

Remark 2.29. Let $\rho : \Gamma_n \to \operatorname{End} A$ be a singular representation of the Kronecker semigroup Γ_n of rank n in $\operatorname{End} A$ such that $\rho(0) = \varphi$, $\varphi \in \operatorname{End} A$, and $\operatorname{rk}(\varphi) = 0$. We can set $\varphi(x_i) = \alpha_i$, $\alpha_i \in K$. Denote by $\psi : K^n \to K^n$ the mapping on K^n such that $\psi(x_1, \dots, x_n) = (x_1 - \alpha_1, \dots, x_n - \alpha_n)$. Define a representation $\hat{\rho} : \Gamma_n \to \operatorname{End} A$ of Γ_n in $\operatorname{End} A$ by the rule $\hat{\rho}(0) = \hat{\rho}(b_{ij}) = \varphi\psi$ for all $i, j \in [1n]$. Then $\varphi\psi = \hat{O}$ and $\hat{\rho}(0) = \hat{O}$, where $\hat{O} \in \operatorname{End} A$ such that $\hat{O}(x_i) = 0$ for all $i \in [1n]$ and $\hat{O}(1) = 1$.

Proposition 2.30. *Let $\rho : \Gamma_n \to \operatorname{End} A$ be a non-singular representation of a Kronecker semigroup Γ_n. Then,* $\operatorname{rk}(\rho(b_{ij})) = 1$ *for all $i, j \in [1n]$.*

Proof. We use the above mentioned relationship (2.2) between endomorphisms $\varphi : K[x_1,\dots,x_n] \to K[x_1,\dots,x_n]$ of the polynomial algebra $K[x_1,\dots,x_n]$ and polynomial maps $\varphi^* = (\varphi_1,\dots,\varphi_n) : K^n \to K^n$ of the affine space K^n into itself, where $\varphi_i(x_1,\dots,x_n) = \varphi(x_i)$.

Denote $\rho(b_{ij})$ by φ_{ij}, $i,j \in [1n]$. Let $\bar\varphi_{ij}$ be the endomorphisms of the algebra $B = K[x_1,\dots,x_n]$ of commutative polynomials in variables $x_1,\dots,x_n$ induced by the endomorphisms φ_{ij} of the algebra A. Clearly, $\bar\varphi_{ij}\bar\varphi_{km} = \delta_{jk}\bar\varphi_{im}$. For a fix $j \in [1n]$ consider $\bar\varphi_{jj}$ as a polynomial mapping from K^n into K^n, i.e., $\bar\varphi_{jj}(x_1,\dots,x_n) = (\bar\varphi_{jj}(x_1),\dots,\bar\varphi_{jj}(x_n))$. Since $\bar\varphi_{jj}^2 = \bar\varphi_{jj}$, the mapping $\bar\varphi_{jj}$ has a fixed point in K^n. This point $d = (d_1,\dots,d_n)$, $d_i \in K$, can be chosen arbitrarily from the image of $\bar\varphi_{jj}$. Therefore, we have $\bar\varphi_{jj}(d_1,\dots,d_n) = (d_1,\dots,d_n)$.

Denote by $T : K^n \to K^n$ the polynomial mapping on K^n such that $T(x_1,\dots,x_n) = (x_1 + d_1,\dots,x_n + d_n)$. Let $\tilde\varphi_{ij} = T^{-1}\bar\varphi_{ij}T$ be a mapping K^n into itself. Denote by $p_{ij}^{(k)}$ the element $T^{-1}\bar\varphi_{ij}T(x_k)$. Since the mapping $\tilde\varphi_{ii}$ has the fixed point $0 \in K^n$, the elements $p_{ii}^{(k)}$ do not have constant terms for any $i,k \in [1n]$. Now we will prove that the elements $p_{ij}^{(k)}$, $i,j,k \in [1n]$, also do not have constant terms. Assume, on the contrary, that there exist $i,j,k \in [1n]$, $i \neq j$, such that the element $p_{ij}^{(k)}$ has a constant term. Since the elements $p_{jj}^{(m)} = T^{-1}\bar\varphi_{jj}T(x_m)$ do not have a constant term for any $m,j \in [1n]$, we obtain

$$\big(T^{-1}\bar\varphi_{jj}T\big)\big(T^{-1}\bar\varphi_{ij}T\big)(x_k) = \big(T^{-1}\bar\varphi_{jj}T\big)p_{ij}^{(k)} \neq 0.$$

On the other hand, since $i \neq j$

$$\big(T^{-1}\bar\varphi_{jj}T\big)\big(T^{-1}\bar\varphi_{ij}T\big)(x_k) = \big(T^{-1}\bar\varphi_{jj}\bar\varphi_{ij}T\big)(x_k) = 0.$$

This contradiction proves that the elements $p_{ij}^{(k)} = T^{-1}\bar\varphi_{ij}T(x_k)$ do not have a constant term for any $i,j,k \in [1n]$. As a consequence, the elements $T^{-1}\varphi_{ij}T(x_k)$ do not have constant terms for any $i,j,k \in [1n]$, too.

Denote the mapping $T^{-1}\varphi_{ij}T : A \to A$ by $\hat\varphi_{ij}$. We now prove that $\hat\varphi_{ij}(A)$ is a subalgebra of $K[w]$ for some $w \in A$. Let I be the ideal of A generated by $x_1,\dots,x_n$. Since the elements $\hat\varphi_{ij}(x_k)$, $i,j,k \in [1n]$, do not have a constant term, $\hat\varphi_{ij}(I^s) \subseteq I^s$ for any $s \geqslant 1$. Now we fix some $i,j \in [1n]$ and consider induced maps $\tilde\varphi_{ij}^{(s)} : I^s/I^{s+1} \to I^s/I^{s+1}$ for any $s \geqslant 1$. We intend to prove that $\operatorname{Im}\tilde\varphi_{ij}^{(s)}$ are one-dimensional vector spaces over K. Let $s = 1$. Then $\tilde\varphi_{ij}^{(1)} : I/I^2 \to I/I^2$ is a linear mapping from the vector space I/I^2 into itself. Since $\tilde\varphi_{ij}^{(1)}\tilde\varphi_{mk}^{(1)} = \delta_{jm}\tilde\varphi_{ik}^{(1)}$, by [11, Lemma 4.7] there exists a basis $\bar z_{r1} = z_r + I^2$, where $z_r \in I$, $r \in [1n]$, of I/I^2 such that $\tilde\varphi_{ij}^{(1)}(\bar z_{r1}) = \delta_{jr}\bar z_{i1}$. For a fix number $s \geqslant 2$ denote $\bar z_{rs} = z_r + I^{s+1}$, $r \in [1n]$. We have $\tilde\varphi_{ij}^{(s)}(\bar z_{i_1 s}\cdots\bar z_{i_s s}) = \delta_{ji_1}\cdots\delta_{ji_s}\bar z_{is}^s$. Thus, $\tilde\varphi_{ij}^{(s)}(I^s/I^{s+1})$ is a one-dimensional vector space with a basis $\{\bar z_{is}^s\}$. The latter assertion holds for any $s \geqslant 2$. As a consequence, we have $\hat\varphi_{ij}(A) \subseteq K[z_i]$. Hence, $\varphi_{ij}(A)$ is a subalgebra of $K[w]$, where $w = Tz_i$. Since the representation ρ of Γ is non-singular, $K \subset \varphi_{ij}(A)$. Thus, $\operatorname{rk}(\varphi_{ij}) = \operatorname{rk}\rho(b_{ij}) = 1$ for all $i,j \in [1n]$. □

2.3. Bases and subbases of the semigroup End A

Definition 2.31. A set of endomorphisms $\mathcal{B}_e = \{e'_{ij} \mid e'_{ij} \in \operatorname{End} A$ and $e'_{ij} \neq \hat O,\ \forall i,j \in [1n]\}$ of A is called a *subbase* of End A if $e'_{ij}e'_{km} = \delta_{jk}e'_{im}$, $\forall i,j,k,m \in [1n]$.

Denote by E' a semigroup of End A generated by endomorphisms e'_{ij} and the endomorphism $\hat O$. By Proposition 2.30, we obtain the following

Corollary 2.32. $\operatorname{rk}(e'_{ij}) = 1$ *for any* $i,j \in [1n]$.

We can assume that $e'_{ij}(A)$ is a subalgebra of $K[z_{ij}]$, $i, j \in [1n]$, where $z_{ij} \in A$. For the sake of simplicity we write $z_{ii} = z_i$, $i \in [1n]$.

Definition 2.33 *("External" definition of a base collection of* End A*).* We say that the subbase $\mathcal{B}_e$ is a *base collection of endomorphisms* of A (or *a base* of End A, for short) if $Z = \{z_i \mid z_i \in A$ such that $e'_{ii}(A) \subseteq K[z_i],\ i \in [1n]\}$ is a base of A.

Now we show that there exists a subbase of End A that is not its base.

Example 2.34. Let $\varphi_{ij} : K[x_1, x_2] \to K[x_1, x_2]$, where $i, j \in \{1, 2\}$, be endomorphisms of the free associative–commutative algebra $A = K[x_1, x_2]$ such that

$$\begin{aligned}
&\varphi_{11}(x_1) = x_1 + x_1x_2, \qquad \varphi_{11}(x_2) = 0, \qquad \varphi_{22}(x_1) = 0, \qquad \varphi_{22}(x_2) = x_2,\\
&\varphi_{12}(x_1) = 0, \qquad \varphi_{12}(x_2) = x_1 + x_1x_2, \qquad \varphi_{21}(x_1) = x_2, \qquad \varphi_{21}(x_2) = 0.
\end{aligned} \tag{2.5}$$

It is easy to see that $\mathrm{rk}(\varphi_{ij}) = 1$ and $\varphi_{ij}\varphi_{km} = \delta_{jk}\varphi_{im}$ for any $i, j, k, m \in \{1, 2\}$, i.e., the set of endomorphisms $B_\varphi = \{\varphi_{ij} \mid \varphi_{ij} \in \operatorname{End} A,\ i, j \in \{1, 2\}\}$ is a subbase of the semigroup End A. We will prove that B_φ is not its base. It is clear that $\varphi_{11}(A) = K[u]$, where $u = x_1 + x_1x_2$, and $\varphi_{22}(A) = K[x_1]$. We can take $z_1 = u$ and $z_2 = x_1$. The elements z_1 and z_2 generate the algebra $K[x_1 + x_1x_2, x_1]$. Let us show that $K[x_1 + x_1x_2, x_2] \neq K[x_1, x_2]$. If, on the contrary, $K[x_1 + x_1, x_2, x_2] = K[x_1, x_2]$ then $x_1 = \alpha(x_1 + x_1x_2) + \beta x_2 + P(u, x_2)$, where $\deg P(u, x_2) \geqslant 2$ and $\alpha, \beta \in K$. Hence $\beta = 0$, $\alpha = 1$ and $P(u, x_2) = 0$. We come to a contradiction. Therefore, the subbase B_φ is not a base of End A.

"Internal" definition of a *base collection* of End A is a bit tricky (see [11,9]). It was inspired by G. Zhitomirski (see [23]).

Definition 2.35 *("Internal" definition of a base collection of* End A*).* The subbase of endomorphisms $\mathcal{B}_e = \{e'_{ij} \mid e'_{ij} \in \operatorname{End} A,\ i, j \in [1n]\}$ of End A is its base if for any collection of endomorphisms $\alpha_i : A \to A$, $\forall i \in [1n]$, and any subbase $\mathcal{B}_f = \{f'_{ij} \mid i, j \in [1n]\}$ of End A there exist endomorphisms $\varphi, \psi \in \operatorname{End} A$ such that

$$\alpha_i \circ f'_{ii} = \psi \circ e'_{ii} \circ \varphi, \quad \text{for all } i \in [1n]. \tag{2.6}$$

Our aim is to prove the statement similar to Proposition 2.27 in [5].

Proposition 2.36. *Internal and external definitions of a base collection of* End A *are equivalent.*

Proof. Let a subbase of endomorphisms $\mathcal{B}_e$ be a base according Definition 2.33. Since $\mathrm{rk}(f'_{ij}) = 1$, $\forall i, j \in [1n]$, there exist elements $y_{ij} \in A$, $i, j \in [1n]$, such that $K \subset f'_{ij}(A(X)) \subseteq K[y_{ij}]$ for all $i, j \in [1n]$. Define endomorphisms ψ and φ of A as follows:

$$\varphi(x_i) = z_i \quad \text{and} \quad \psi(z_i) = \alpha_i(y_i), \quad \text{for all } i \in [1n],$$

where $e'_{ii}(A) \subseteq K[z_i]$, $z_i \in A$, and $y_i = y_{ii}$, $\forall i \in [1n]$. Since $Z = \langle z_i \mid z_i \in A,\ i \in [1n]\rangle$ is a base of A, the endomorphism ψ is well defined. Now it is easy to check that the condition (2.6) with the given φ and ψ is fulfilled.

Conversely, assume that the condition (2.6) is fulfilled for the subbase $\mathcal{B}_e$. Let us prove that $Z = \langle z_i \mid z_i \in A,\ i \in [1n]\rangle$ is a base of A. Choosing $\alpha_i = e_{ii}$ and $f'_{ij} = e_{ij}$, $i, j \in [1n]$, in (2.6), we obtain

$$e_{ii} = \psi \circ e'_{ii} \circ \varphi,$$

i.e., $\psi(e'_{ii}\varphi(x_i)) = x_i$ for any $i \in [1n]$. Denote by $t_i = e'_{ii}\varphi(x_i)$. We have $\psi(t_i) = x_i$. Since A is Hopfian, i.e., any surjective endomorphism of A into itself is isomorphism, the elements t_i, $i \in [1n]$, form the base of A. By Corollary 2.32 and Remark 2.22, $K \subset e'_{ii}(A) \subseteq K[z_i]$. Therefore, there exists a non-scalar polynomial $\chi_i(z_i) \in K[z_i]$ such that $t_i = \chi_i(z_i)$. Since $t_i = \chi_i(z_i)$, $i = 1, \ldots, n$, forms the base of A, the elements z_i, $i = 1, \ldots, n$, form a base of A as claimed. □

Now we deduce

Corollary 2.37. *Let $\Phi \in \operatorname{Aut}\operatorname{End} A$ and E be the subsemigroup of* End A *generated by the Kronecker endomorphisms e_{ij}, $i, j \in [1n]$ (see Definition* 2.23*). Then $\mathcal{C} = \{\Phi(e_{ij}) \mid i, j \in [1n]\}$ is a base of* End A.

Proof. Assume that $\mathrm{rk}(\Phi(e_{ij})) = 0$ for some $i, j \in [1n]$. By Corollary 2.21, we obtain $\mathrm{rk}(e_{ij}) = 0$. We arrived at a contradiction. Thus, $\mathrm{rk}(\Phi(e_{ij})) \neq 0$. Since $\Phi(e_{ij})\Phi(e_{km}) = \delta_{jk}\Phi(e_{im})$, the set $\mathcal{C}$ is a subbase of End A. It is easy to check that the condition (2.6) is fulfilled for the subbase $\mathcal{C}$. Thus, $\mathcal{C}$ is a base of End A. □

Lemma 2.38. *Let $\mathcal{B}_e = \{e'_{ij} \mid e'_{ij} \in \operatorname{End} A,\ i, j \in [1n]\}$ be a base collection of endomorphisms of* End A. *Then there exists a base $Z' = \{z'_k \mid z'_k \in A,\ k \in [1n]\}$ of A such that the endomorphisms e'_{ij} from $\mathcal{B}_e$ are Kronecker ones of A in Z'.*

Proof. With the preceding notation from Definition 2.33 we have that the equality $(e'_{ii})^2 = e'_{ii}$ implies $e'_{ii}(z_i) = z_i$, $i \in [1n]$. Since $e'_{ii}e'_{ij}(z_j) = e'_{ij}(z_j)$ and $K \subset e'_{ii}(A) \subseteq K[z_i]$, there exists a non-scalar polynomial $f_j(z_i) \in K[z_i]$ such that $e'_{ij}(z_j) = f_j(z_i)$. Similarly, there exists a non-scalar polynomial $g_i(z_j) \in K[z_j]$ such that $e'_{ji}(z_i) = g_i(z_j)$. We have

$$z_j = e'_{jj}(z_j) = e'_{ji}e'_{ij}(z_j) = e'_{ji}\bigl(f_j(z_i)\bigr) = f_j\bigl(g_i(z_j)\bigr) \quad \text{for all } i, j \in [1n]$$

and, in a similar way, $z_i = g_i(f_j(z_i))$ for all $i, j \in [1n]$. Thus f_j and g_i are linear polynomials over K in variables z_i and z_j, respectively. Therefore,

$$e'_{ij}(z_j) = a_i z_i + b_i, \quad a_i, b_i \in K \text{ and } a_i \neq 0. \tag{2.7}$$

Note that $e'_{ij}(z_k) = e'_{ij}(e'_{kk}(z_k)) = 0$ if $k \neq j$. Now we have for $i \neq j$

$$0 = {e'_{ij}}^2(z_j) = e'_{ij}(a_i z_i + b_i) = e'_{ij}(b_i) = b_i,$$

i.e., $e'_{ij}(z_j) = a_i z_i$, $a_i \neq 0$. Let $z'_i = a_i^{-1} z_i$. We obtain a base $Z = \{z'_k \mid z'_k \in A,\ k \in [1n]\}$ of A such that $e'_{ij}(z'_k) = \delta_{jk} z'_k$, $i, j, k \in [1n]$, i.e., e'_{ij} are Kronecker endomorphisms of A in the base Z'. The proof is completed. □

3. Automorphisms of the semigroup End A

3.1. *On the group* Aut End A

We need the following notion.

Definition 3.1. (See [7].) Let A_1 and A_2 be algebras over K from a variety $\mathcal{A}$, δ be an automorphism of K and $\varphi: A_1 \to A_2$ be a ring homomorphism of these algebras. A pair (δ, φ) is called a *semi-linear homomorphism* from A_1 to A_2 if

$$\varphi(\alpha \cdot u) = \delta(\alpha) \cdot \varphi(u), \quad \forall \alpha \in K,\ \forall u \in A_1.$$

Definition 3.2. (See [17].) An automorphism Φ of the semigroup $\mathrm{End}\, A$ of endomorphisms of A is called *quasi-inner* if there exists an *adjoined bijection* $s : A \to A$ such that $\Phi(\nu) = s\nu s^{-1}$, for any $\nu \in \mathrm{End}\, A$.

Definition 3.3. (See [17].) A quasi-inner automorphism Φ of $\mathrm{End}\, A$ is called *semi-inner* if there exists a field automorphism $\delta : K \to K$ such that (δ, s) is a semi-linear automorphism of A, i.e., for any $\alpha \in K$ and $a, b \in A$ the following conditions hold:

1. $s(a + b) = s(a) + s(b)$,
2. $s(a \cdot b) = s(a) \cdot s(b)$,
3. $s(\alpha a) = \delta(\alpha)s(a)$.

We say that the pair (δ, s) defines the semi-inner automorphism Φ of A with the *adjoined ring automorphism* s. If δ is the identity automorphism of K, we call the automorphism Φ *inner*.

The description of quasi-inner automorphisms of $\mathrm{End}\, A$ is as follows.

Proposition 3.4. *(See [3,9,11].) Let $\Phi \in \mathrm{Aut}\,\mathrm{End}\, A$ be a quasi-inner automorphism of* $\mathrm{End}\, A$*. Then Φ is of semi-inner automorphisms of* $\mathrm{End}\, A$.

We will use the following fact:

Proposition 3.5. *(See [9,11].) Let $\Phi \in \mathrm{Aut}\,\mathrm{End}\, A$ and E be the subsemigroup of* $\mathrm{End}\, A$ *generated by e_{ij}, $i, j \in [1n]$. Elements of the semigroup $\Phi(E)$ are Kronecker endomorphisms of A in some base $U = \{u_1, \dots, u_n\}$, $u_i \in A$, if and only if Φ is a quasi-inner automorphism of* $\mathrm{End}\, A$.

Now we obtain one of the main results of the paper.

Theorem 3.6. *Every automorphism of the group* $\mathrm{Aut}\,\mathrm{End}\, A$ *is semi-inner.*

Proof. By Corollary 2.37, the set of endomorphisms $\mathcal{C} = \{\Phi(e_{ij}) \mid \forall i \in [1n]\}$ is a base collection of endomorphisms of A. By Lemma 2.38, there exists a base $S = \langle s_k \mid s_k \in A,\ k \in [1n]\rangle$ such that the endomorphisms $\Phi(e_{ij})$ are Kronecker endomorphisms in S. According to Proposition 3.5, we obtain that Φ is quasi-inner. By virtue of Proposition 3.4, every automorphism of the group $\mathrm{Aut}\,\mathrm{End}\, A$ is semi-inner and as claimed. $\square$

Remark 3.7. If $\mathcal{CA}$ is the category of commutative–associative algebras over a field K, let $\mathcal{SCA}$ be the category with the same objects as in the category $\mathcal{CA}$, morphisms be all pairs $\psi_\delta = (\psi, \delta) : A \to B$, $A, B \in \mathrm{Ob}\,\mathcal{SCA}$, such that $\psi : A \to B$ are ring homomorphisms from A to B, $\delta : K \to K$ are automorphisms of the field K and $\psi_\delta(\lambda a) = \lambda^\delta \psi(a)$, $a \in A$. Morphisms ψ_δ of the category $\mathcal{SCA}$ are called *semi-linear homomorphisms* (or *semi-homomorphisms*) from A to B (cf. Definition 3.1). Denote by $\mathrm{SEnd}\, A$ the semigroup of semi-endomorphisms of A with the usual composition of maps in the category $\mathcal{SCA}$.

Clearly, that the definitions of endomorphisms of rank 1 and 0 can be transfer to the category $\mathcal{SCA}$. All results about bases and subbases from Section 2.3 are also true. As a consequence, we obtain the following

Theorem 3.8. *Every automorphism of the group* $\mathrm{Aut}\,\mathrm{SEnd}\, A$ *is semi-inner.*

4. Automorphisms of the category $\mathcal{A}^{\circ}$

Recall the following notions of the category isomorphism and equivalence (cf. [12]). *An isomorphism $\varphi : \mathcal{C} \to \mathcal{M}$ of categories* is a functor φ from $\mathcal{C}$ to $\mathcal{M}$, which is a bijection both on objects and morphisms. In other words, there exists a functor $\psi : \mathcal{M} \to \mathcal{C}$ such that $\psi\varphi = 1_{\mathcal{C}}$ and $\varphi\psi = 1_{\mathcal{M}}$.

Let φ_1 and φ_2 be two functors from $\mathcal{C}_1$ to $\mathcal{C}_2$. *A functor isomorphism $s : \varphi_1 \to \varphi_2$* is a collection of isomorphisms $s_D : \varphi_1(D) \to \varphi_2(D)$ defined for all $D \in \mathrm{Ob}\,\mathcal{C}_1$ such that for every $\nu : D \to B$, $\nu \in \mathrm{Mor}\,\mathcal{C}_1$, $B \in \mathrm{Ob}\,\mathcal{C}_1$

$$s_B \cdot \varphi_1(\nu) = \varphi_2(\nu) \cdot s_D$$

holds, i.e., the following diagram

$$\begin{tikzcd} \varphi_1(D) \arrow[r, "s_D"] \arrow[d, "\varphi_1(\nu)"'] & \varphi_2(D) \arrow[d, "\varphi_2(\nu)"] \\ \varphi_1(B) \arrow[r, "s_B"] & \varphi_2(B) \end{tikzcd}$$

is commutative. An isomorphism of functors φ_1 and φ_2 is denoted by $\varphi_1 \cong \varphi_2$.

An equivalence of categories $\mathcal{C}$ and $\mathcal{M}$ is a pair of functors $\varphi : \mathcal{C} \to \mathcal{M}$ and $\psi : \mathcal{M} \to \mathcal{C}$ such that $\psi\varphi \cong 1_{\mathcal{C}}$ and $\varphi\psi \cong 1_{\mathcal{M}}$. If $\mathcal{C} = \mathcal{M}$, then we get the notions of *automorphism* and *autoequivalence* of the category $\mathcal{C}$.

For every small category $\mathcal{C}$, denote the group of all its automorphisms by $\mathrm{Aut}\,\mathcal{C}$. We distinguish the following classes of automorphisms of $\mathcal{C}$.

Definition 4.1. (See [8,15,20].) An automorphism $\varphi : \mathcal{C} \to \mathcal{C}$ is *equinumerous* if $\varphi(D) \cong D$ for any object $D \in \mathrm{Ob}\,\mathcal{C}$; φ is *stable* if $\varphi(D) = D$ for any object $D \in \mathrm{Ob}\,\mathcal{C}$; and φ is *inner* if φ and $1_{\mathcal{C}}$ are naturally isomorphic, i.e., $\varphi \cong 1_{\mathcal{C}}$.

In other words, an automorphism φ is inner if for all $D \in \mathrm{Ob}\,\mathcal{C}$ there exists an isomorphism $s_D : A \to \varphi(D)$ such that

$$\varphi(\nu) = s_B \nu s_D^{-1} : \varphi(D) \to \varphi(B)$$

for any morphism $\nu \in \mathrm{Mor}_{\mathcal{C}}(A, B)$.

Denote by $\mathrm{Eqn}\,\mathrm{Aut}\,\mathcal{C}$, $\mathrm{St}\,\mathrm{Aut}\,\mathcal{C}$, and $\mathrm{Int}\,\mathcal{C}$ the collections of equinumerous, stable, and inner automorphisms of the group $\mathrm{Aut}\,\mathcal{C}$, respectively.

Let Θ be a variety of linear algebras over K. Denote by Θ^0 *the full subcategory of finitely generated free algebras* $F(X)$, $|X| < \infty$, of the variety Θ. Consider a constant morphism $\nu_0 : F(X) \to F(X)$ such that $\nu_0(x) = x_0$, $x_0 \in F(X)$, for every $x \in X$.

Theorem 4.2 *(Reduction Theorem). (See [8,13,16,20,23].) Let the free algebra $F(X)$ generate a variety Θ, and $\varphi \in \mathrm{St}\,\mathrm{Aut}\,\Theta^0$. If φ acts trivially on the monoid $\mathrm{Mor}_{\Theta^0}(F(X), F(X))$ and $\varphi(\nu_0) = \nu_0$, then φ is inner, i.e., $\varphi \in \mathrm{Int}\,\Theta^0$.*

Define the notion of a semi-inner automorphism of the category Θ^0 of free finitely generated algebras in the category Θ.

Definition 4.3. (See [15].) An automorphism $\varphi \in \mathrm{Aut}\,\Theta^0$ is called *semi-inner* if there exists a family of semi-isomorphisms $\{s_{F(X)} = (\delta, \tilde{\varphi})\colon\ F(X) \to \tilde{\varphi}(F(X)),\ F(X) \in \mathrm{Ob}\,\Theta^0\}$, where $\delta \in \mathrm{Aut}\,K$ and $\tilde{\varphi}$ is a ring isomorphism from $F(X)$ to $\tilde{\varphi}(F(X))$ such that for any homomorphism $\nu : F(X) \to F(Y)$ the following diagram

$$\begin{tikzcd} F(X) \arrow[r, "s_{F(X)}"] \arrow[d, "\nu"'] & \tilde{\varphi}(F(X)) \arrow[d, "\varphi(\nu)"] \\ F(Y) \arrow[r, "s_{F(Y)}"] & \tilde{\varphi}(F(Y)) \end{tikzcd}$$

is commutative.

Further, we will need the following

Proposition 4.4. *(See [8,15].) For any equinumerous automorphism $\varphi \in \mathrm{Aut}\,\mathcal{C}$ there exist a stable automorphism φ_S and an inner automorphism φ_I of the category $\mathcal{C}$ such that $\varphi = \varphi_S \varphi_I$.*

Now we give a description of the groups $\mathrm{Aut}\,\mathcal{C}\mathcal{A}^\circ$ over any field. Note that a description of this group over infinite fields was given in [2].

Theorem 4.5. *All automorphisms of the group $\mathrm{Aut}\,\mathcal{A}^\circ$ of automorphisms of the category $\mathcal{C}\mathcal{A}^\circ$ are semi-inner automorphisms of the category $\mathcal{C}\mathcal{A}^\circ$.*

Proof. Let $\varphi \in \mathrm{Aut}\,\mathcal{A}^\circ$. It is clear that φ is an equinumerous automorphism. By Proposition 4.4, φ can be represented as a composition of a stable automorphism φ_S and an inner automorphism φ_I. Since stable automorphisms do not change free algebras from $\mathcal{A}^\circ$, we obtain that $\varphi_S \in \mathrm{Aut}\,\mathrm{End}\,A$. By Theorem 3.6, φ_S is semi-inner of $\mathrm{End}\,A$. Using this fact and Reduction Theorem 4.2, we obtain that all automorphisms of the group $\mathrm{Aut}\,\mathcal{C}\mathcal{A}^\circ$ are semi-inner automorphisms of the category $\mathcal{C}\mathcal{A}^\circ$. This completes the proof. □

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