Journal of Philosophical Logic (2019) 48:787–808
https://doi.org/10.1007/s10992-018-9494-x

The FMLA-FMLA Axiomatizations of the Exactly True and Non-falsity Logics and Some of Their Cousins

Yaroslav Shramko[1] · Dmitry Zaitsev[2] · Alexander Belikov[2]

Received: 21 March 2018 / Accepted: 2 November 2018 / Published online: 10 November 2018

Abstract

In this paper we present a solution of the axiomatization problem for the FMLA-FMLA versions of the Pietz and Rivieccio exactly true logic and the non-falsity logic dual to it. To prove the completeness of the corresponding binary consequence systems we introduce a specific proof-theoretic formalism, which allows us to deal simultaneously with two consequence relations within one logical system. These relations are hierarchically organized, so that one of them is treated as the basic for the resulting logic, and the other is introduced as an extension of this basic relation. The proposed bi-consequences systems allow for a standard Henkin-style canonical model used in the completeness proof. The deductive equivalence of these bi-consequence systems to the corresponding binary consequence systems is proved. We also outline a family of the bi-consequence systems generated on the basis of the first-degree entailment logic up to the classic consequence.

Keywords First-degree entailment · Exactly true logic · Non-falsity logic · FMLA-FMLA logical framework · Bi-consequence system

✉ Yaroslav Shramko
shramko@rocketmail.com

Dmitry Zaitsev
zaitsev@philos.msu.ru

Alexander Belikov
belikov_sasha@bk.ru

[1] Department of Philosophy, Kryvyi Rih State Pedagogical University, 50086 Kryvyi Rih, Ukraine

[2] Department of Logic, Lomonosov Moscow State University, 119899 Moscow, Russia

1 Preliminaries: Dunn-Belnap's Four-Valued Semantics in Different Logical Settings

Michael Dunn and Nuel Belnap's four-valued logic rang in its 40th anniversary[1] as a well established research area with a vast number of topics on various aspects of its syntactic and semantic characterizations. Among these topics is also the topic of *designated* values, selection of which from the Belnapian set of "true only" (T), "false only" (F), "both true and false" (B) and "neither true nor false" (N) plays a key role in defining the entailment relation. There seems to be a broad consensus about such a selection, based normally on Belnap's guideline that "the inference from A to B is valid, or that A entails B, if the inference never leads us from told True to the absence of told True (preserves Truth), and also never leads us from the absence of told False to told False (preserves non-Falsity)" [4, p. 519]. Here "told True" means that a sentence is either true only or both true and false (is *at least true*), and analogously for "told False".

Consider sentential language $\mathcal{L}$ defined as follows:

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid {\sim}\varphi.$$

Let a generalized valuation v be a map from the set of sentential variables to the *subsets* of the set of classical truth-values $\{t, f\}$, cf. [9, p. 156]. This valuation is extended to the whole language by the following conditions:

Definition 1

(1) $t \in v(\varphi \wedge \psi) \Leftrightarrow t \in v(\varphi)$ *and* $t \in v(\psi)$,
$f \in v(\varphi \wedge \psi) \Leftrightarrow f \in v(\varphi)$ *or* $f \in v(\psi)$;
(2) $t \in v(\varphi \vee \psi) \Leftrightarrow t \in v(\varphi)$ *or* $t \in v(\psi)$,
$f \in v(\varphi \vee \psi) \Leftrightarrow f \in v(\varphi)$ *and* $f \in v(\psi)$;
(3) $t \in v({\sim}\varphi) \Leftrightarrow f \in v(\varphi)$,
$f \in v({\sim}\varphi) \Leftrightarrow t \in v(\varphi)$.

Belnap's four truth values (being ascribed to a sentence φ) are then explicated as follows:

$$v(\varphi) = B \Leftrightarrow t \in v(\varphi) \text{ and } f \in v(\varphi),$$
$$v(\varphi) = T \Leftrightarrow t \in v(\varphi) \text{ and } f \notin v(\varphi),$$
$$v(\varphi) = F \Leftrightarrow t \notin v(\varphi) \text{ and } f \in v(\varphi),$$
$$v(\varphi) = N \Leftrightarrow t \notin v(\varphi) \text{ and } f \notin v(\varphi).$$

Josep Maria Font in [15, p. 5] explicates "Belnap's logic" semantically by an entailment relation $\vDash_{\mathcal{B}}$ between arbitrary *sets* of sentences (Γ) and single sentences (ψ). In terms of generalized valuation v, it can be defined as follows, cf. also [26, p. 144]:

[1] This is an approximate date marked by the appearance in 1976–1977 of the seminal papers [6, 7, 9], see in particular the special issue of *Studia Logica* "40 years of FDE" [24].

Definition 2 $\Gamma \vDash_{\mathcal{B}} \psi =_{df} \forall v : (\forall \varphi \in \Gamma : t \in v(\varphi)) \Rightarrow t \in v(\psi)$.

This definition is based on the first half of Belnap's guideline for entailment cited above, taking the set of designated truth values to be $\{T, B\}$.[2]

Font presents in [15, p. 10] a proof system $\vdash_H$ that determines the set of valid consequences of the form $\Gamma \vdash \psi$ by the following direct inference rules:

$$\text{(R1)}\ \frac{\varphi \wedge \psi}{\varphi} \qquad \text{(R2)}\ \frac{\varphi \wedge \psi}{\psi} \qquad \text{(R3)}\ \frac{\varphi, \psi}{\varphi \wedge \psi}$$

$$\text{(R4)}\ \frac{\varphi}{\varphi \vee \psi} \qquad \text{(R5)}\ \frac{\varphi \vee \psi}{\psi \vee \varphi} \qquad \text{(R6)}\ \frac{\varphi \vee \varphi}{\varphi}$$

$$\text{(R7)}\ \frac{\varphi \vee (\psi \vee \chi)}{(\varphi \vee \psi) \vee \chi} \qquad \text{(R8)}\ \frac{\varphi \vee (\psi \wedge \chi)}{(\varphi \vee \psi) \wedge (\varphi \vee \chi)} \qquad \text{(R9)}\ \frac{(\varphi \vee \psi) \wedge (\varphi \vee \chi)}{\varphi \vee (\psi \wedge \chi)}$$

$$\text{(R10)}\ \frac{\varphi \vee \psi}{\sim\sim\varphi \vee \psi} \qquad \text{(R11)}\ \frac{\sim\sim\varphi \vee \psi}{\varphi \vee \psi} \qquad \text{(R12)}\ \frac{\sim(\varphi \vee \psi) \vee \chi}{(\sim\varphi \wedge \sim\psi) \vee \chi}$$

$$\text{(R13)}\ \frac{(\sim\varphi \wedge \sim\psi) \vee \chi}{\sim(\varphi \vee \psi) \vee \chi} \qquad \text{(R14)}\ \frac{\sim(\varphi \wedge \psi) \vee \chi}{(\sim\varphi \vee \sim\psi) \vee \chi} \qquad \text{(R15)}\ \frac{(\sim\varphi \vee \sim\psi) \vee \chi}{\sim(\varphi \wedge \psi) \vee \chi}$$

Font describes $\vdash_H$ as a "Hilbert-style presentation" of Belnap's logic, and proves its soundness and completeness with respect to Definition 2.

It is often ignored, however, that—in an important aspect—the notion of "Belnap's logic" promoted by Font *is not genuinely Belnapian*. As Font himself remarks [15, p. 5], his presentation of this logic is in fact an "extension" of Belnap's original idea from [6], where entailment has been semantically defined as a relation *between (single) sentences*. Dunn in [9] also deals with entailment as a relation that "holds relevantly between truth-functional *sentences*" (p. 149, emphasis added). By using the generalized valuation v this relation can be defined as follows:

Definition 3 $\varphi \vDash_{\mathcal{B}_1} \psi =_{df} \forall v : t \in v(\varphi) \Rightarrow t \in v(\psi)$.

Originally, Dunn-Belnap's four-valued framework was designed to provide a semantic modeling for Anderson and Belnap's system of *first-degree entailment* from [3, § 15]. Again, in the strict sense, first-degree entailments are implicational expressions of the form $\varphi \to \psi$, where φ and ψ can be "truth functions of any degree but cannot contain any arrows" [3, p. 150]. Currently it is more common to employ the *binary consequence expressions* of the form $\varphi \vdash \psi$ (to be read as "φ has ψ as a consequence" [10, p. 302]), where $\varphi, \psi \in \mathcal{L}$. The corresponding proof systems (called "binary consequence systems" by Dunn [11, p. 24], and "symmetric consequence systems" by Chrysafis Hartonas [17, p. 5]) manipulate binary consequences as formal objects. Such systems are of interest in their own right, as an important particular way of presenting logical structures.

[2] Belnap observes: "Dunn 1975 has shown that it suffices to mention truth-preservation, since if some inference form fails to always preserve non-Falsity, then it can be shown by a technical argument that it also fails to preserve Truth" [7, p. 43].

Dunn and Hardegree in [13, p. 185] differentiate between four such presentations:

(1) unary assertional systems, $\vdash \phi$;
(2) binary implicational systems, $\phi \vdash \psi$;
(3) asymmetric consequence systems, $\Gamma \vdash \phi$;
(4) symmetric consequence systems, $\Gamma \vdash \Delta$.

It is observed that (1) can be viewed as a special case of (3), and (3) is a special case of (4), whereas (2) is a special case of both (3) and (4). They also remark that "binary implicational systems are perhaps the presentation that most fits the idea of thinking of logics as ordered algebras" [13, p. 186].

In a similar vein, Lloyd Humberstone [18] elaborated on the idea of *logical frameworks* as specific structures for manipulating *sequents* of various kinds. A particular logical framework assigns to each language a class of sequents permissible within this framework, cf. [18, p. 103]. For example, the logical framework SET-FMLA, "takes a sequent ... to have the form $\Gamma \succ B$ where Γ is a finite (possibly empty) set of formulas ... and B is a formula" [18, p. 103].[3]

Humberstone's classification of logical frameworks includes a separate category for the FMLA-FMLA sequents, considered to be "a suitable setting in which to concentrate on entailment as a binary relation between formulas" [18, p. 108]. In full agreement with the above-cited remark by Dunn and Hardegree, Humberstone [18, p. 246] explains how one can naturally design adequate algebraic semantics for the FMLA-FMLA sequents, based on the relation of pre-order $\leq$ (which can also be restricted to a partial order if needed). In terms of consequences, if one defines a homomorphism h from a given language to a set of propositions, then a binary consequence $\varphi \vdash \psi$ is said to *hold* on this h when $h(\varphi) \leq h(\psi)$, and it is said to be *valid* when it holds on every such homomorphism. It is worth noting that the underlying set of propositions is usually taken to form a *lattice*, which enables us to have conjunction and disjunction in our language.[4] Thus, the FMLA-FMLA consequence systems may play an important role in determining the corresponding algebraic structures.

Now we return to Anderson and Belnap's logic of first-degree entailment. It is axiomatized in [3, p. 158] by the following FMLA-FMLA proof system:

$(a1)$ $\varphi \wedge \psi \vdash \varphi$
$(a2)$ $\varphi \wedge \psi \vdash \psi$
$(a3)$ $\varphi \vdash \varphi \vee \psi$
$(a4)$ $\psi \vdash \varphi \vee \psi$
$(a5)$ $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee \chi$
$(a6)$ $\varphi \vdash \sim\sim\varphi$
$(a7)$ $\sim\sim\varphi \vdash \varphi$

$(r1)$ $\varphi \vdash \psi,\ \psi \vdash \chi \ /\ \varphi \vdash \chi$

[3]Here $\succ$ is a special symbol that "combines formulas into sequents", used by Humberstone as a "sequent separator". Because we deal with consequence expressions, we follow Dunn by employing the sign of a consequence relation $\vdash$ in this place.

[4]On two ways of semantic presentation of various many-valued logics (the four-valued Dunn-Belnap logic among them) see in [38] for more detail.

$(r2)$ $\varphi \vdash \psi, \varphi \vdash \chi \,/\, \varphi \vdash \psi \wedge \chi$
$(r3)$ $\varphi \vdash \chi, \psi \vdash \chi \,/\, \varphi \vee \psi \vdash \chi$
$(r4)$ $\varphi \vdash \psi \,/\, {\sim}\psi \vdash {\sim}\varphi$

In [3] this system is called $\mathbf{E}_{\text{fde}}$, reflecting the fact that it constitutes the first-degree entailment fragment of the calculus **E**. Another (deductively equivalent) axiomatization of Dunn-Belnap's semantics is presented in [12, p. 12] under the label $\mathbf{R}_{\text{fde}}$, stressing thus the point that the first-degree entailment fragments of the systems **E** and **R** are the same.[5] System $\mathbf{R}_{\text{fde}}$ is obtained from $\mathbf{E}_{\text{fde}}$ by excluding $(r4)$ from the inference rules and accepting instead the four De Morgan laws as axioms:

(dm_1) ${\sim}(\varphi \vee \psi) \vdash {\sim}\varphi \wedge {\sim}\psi$
(dm_2) ${\sim}\varphi \wedge {\sim}\psi \vdash {\sim}(\varphi \vee \psi)$
(dm_3) ${\sim}(\varphi \wedge \psi) \vdash {\sim}\varphi \vee {\sim}\psi$
(dm_4) ${\sim}\varphi \vee {\sim}\psi \vdash {\sim}(\varphi \wedge \psi)$.

Whereas (dm_1)–(dm_4) are derivable in $\mathbf{E}_{\text{fde}}$, the rule $(r4)$ is not derivable in $\mathbf{R}_{\text{fde}}$, although it remains admissible in it, see [12, Proposition 11]. Both $\mathbf{E}_{\text{fde}}$ and $\mathbf{R}_{\text{fde}}$ are sound and complete with respect to Definition 3, see [12, Theorem 7].

The relationship between the SET-FMLA and FMLA-FMLA presentations of Belnap's logic is rather straightforward, in that $\Gamma \models_{\mathcal{B}} \psi$ if and only if there is a finite set $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$, such that $\varphi_1 \wedge \ldots \wedge \varphi_n \models_{\mathcal{B}_1} \psi$. Nevertheless, system $\vdash_H$ is much more "flexible" than $\mathbf{E}_{\text{fde}}$ and $\mathbf{R}_{\text{fde}}$ in terms of receptivity to possible extensions. Some of such extensions, connected to a new choice of designated truth values will be considered in the next section.

2 Nothing But the Truth, Anything but Falsehood and Consequences of the FMLA-FMLA Type

Observe once again that both $\models_{\mathcal{B}}$ and $\models_{\mathcal{B}_1}$ defined by Definitions 2 and 3 respectively, imply acceptance of $\{T, B\}$ as the set of designated truth values. However, there may well be some other options for picking out designated elements from the Belnapian set. Arnon Avron in [5, p. 142–143] mentions two possible classes of four-valued logics with the sets of designated truth values $\{T\}$, and $\{T, B, N\}$. He observes a duality between these classes, and hints at a method of constructing the corresponding Gentzen-type systems. João Marcos in [22] develops a uniform semantic approach to entailment relations based on different subsets of designated truth values from $\{N, F, T, B\}$ the subsets $\{T\}$ and $\{N, T, B\}$ among them. He formulates semantic constructions in terms of only two classical-like truth values, and naturally extracts from them the two-signed tableau systems for characterizing the corresponding entailment relations.

[5] A *first-degree* entailment interpretation of the binary consequence systems reveals their important logical role as possible bases for more extended logical systems possessing a nested implication with certain properties.

More recently, Andreas Pietz[6] and Umberto Rivieccio investigate in [25] a logic based on the four Belnapian truth values, but with T as the only designated element. They strengthen Belnap's conditions of truth-preservation and non-falsity-preservation by gluing them together, and demand that a consequence relation must preserve "truth-and-non-falsity" *en bloc*, arriving thus at what they call *exactly true logic*—**ETL**.

Pietz and Rivieccio conceive their exactly true logic as an extension of "Belnap's logic" as presented by Font in [15], that is, dealing with an entailment relation of the SET-FMLA type $\models_{\mathcal{T}}$ (symbolics adjusted) defined as follows:

Definition 4 $\Gamma \models_{\mathcal{T}} \psi =_{df} \forall v : (\forall \varphi \in \Gamma : v(\varphi) = T) \Rightarrow v(\psi) = T.$

Pietz and Rivieccio show how one can obtain a proof-theoretic characterization of this relation by extending Font's system $\vdash_H$ with Ackermann's rule γ, see [2, p. 119]:[7]

$$\frac{\varphi \wedge (\sim\varphi \vee \psi)}{\psi}$$

The resulting system is called **ETL**, and it has been shown to be sound and complete with respect to Definition 4, see [25, Theorem 3.4].

In [35] we consider an entailment relation $\models_{\mathcal{F}}$, which dually to $\models_{\mathcal{T}}$ is based on the set of designated truth values $\{N, T, B\}$ (cf. the above observation by Avron). The mentioned duality concerns also the structure of the resulting relation, defined as a relation between single formulas and some sets of formulas:

Definition 5 $\varphi \models_{\mathcal{F}} \Delta =_{df} \forall v : (\forall \psi \in \Delta : v(\psi) = F) \Rightarrow v(\varphi) = F.$

The characteristic inference rule for $\models_{\mathcal{F}}$ is the *dual* γ:[8]

$$\frac{\varphi}{\psi \vee (\sim\psi \wedge \varphi)}$$

This rule is added to a certain dualization of Font's system $\vdash_{dH}$:

$$(\mathrm{R1}_d)\ \frac{\varphi}{\varphi \vee \psi} \qquad (\mathrm{R2}_d)\ \frac{\psi}{\varphi \vee \psi} \qquad (\mathrm{R3}_d)\ \frac{\varphi \vee \psi}{\varphi, \psi}$$

$$(\mathrm{R4}_d)\ \frac{\varphi \wedge \psi}{\varphi} \qquad (\mathrm{R5}_d)\ \frac{\varphi \wedge \psi}{\psi \wedge \varphi} \qquad (\mathrm{R6}_d)\ \frac{\varphi}{\varphi \wedge \varphi}$$

$$(\mathrm{R7}_d)\ \frac{(\varphi \wedge \psi) \wedge \chi}{\varphi \wedge (\psi \wedge \chi)} \qquad (\mathrm{R8}_d)\ \frac{(\varphi \wedge \psi) \vee (\varphi \wedge \chi)}{\varphi \wedge (\psi \vee \chi)} \qquad (\mathrm{R9}_d)\ \frac{\varphi \wedge (\psi \vee \chi)}{(\varphi \wedge \psi) \vee (\varphi \wedge \chi)}$$

$$(\mathrm{R10}_d)\ \frac{\sim\sim\varphi \wedge \psi}{\varphi \wedge \psi} \qquad (\mathrm{R11}_d)\ \frac{\varphi \wedge \psi}{\sim\sim\varphi \wedge \psi} \qquad (\mathrm{R12}_d)\ \frac{(\sim\varphi \vee \sim\psi) \wedge \chi}{\sim(\varphi \wedge \psi) \wedge \chi}$$

$$(\mathrm{R13}_d)\ \frac{\sim(\varphi \wedge \psi) \wedge \chi}{(\sim\varphi \vee \sim\psi) \wedge \chi} \qquad (\mathrm{R14}_d)\ \frac{(\sim\varphi \wedge \sim\psi) \wedge \chi}{\sim(\varphi \vee \psi) \wedge \chi} \qquad (\mathrm{R15}_d)\ \frac{\sim(\varphi \vee \psi) \wedge \chi}{(\sim\varphi \wedge \sim\psi) \wedge \chi},$$

[6] After the name change—Andreas Kapsner.

[7] In [25] this rule is called "disjunctive syllogism", which is not quite accurate.

[8] In [35] we used the rule of *dual disjunctive syllogism* in the form $\frac{\varphi}{\sim\psi \vee (\psi \wedge \varphi)}$.

producing thus what can be called the "non-falsity logic"—**NFL**, which is sound and complete with respect to Definition 5, see [31].[9]

It should be stressed that both $\vDash_{\mathcal{T}}$ and $\vDash_{\mathcal{F}}$ are defined as *asymmetric* entailment relations (just like $\vDash_{\mathcal{B}}$ and $\vDash_{\mathcal{DB}}$). In particular, Pietz and Rivieccio do not consider the logic of first-degree entailment constructed through the binary implicational systems. Such systems, however, may prove useful for certain logic and algebraic purposes. For example, they might serve as the basis for systems with *nested* exactly true and non-false implications, and moreover, they could contribute to elucidation of the nature of the underlying ordered structures. Thus, the FMLA-FMLA versions of $\vDash_{\mathcal{T}}$ and $\vDash_{\mathcal{F}}$ deserve special attention.

In [35, p. 1314] we set up the problem of deductive formalizations of the exactly true logic of the FMLA-FMLA type and its dual by means of the binary (symmetric) consequence systems. Namely, consider the following straightforward definitions (for any $\varphi, \psi \in \mathcal{L}$):

Definition 7 $\varphi \vDash_{\mathcal{T}_1} \psi =_{df} \forall v : v(\varphi) = T \Rightarrow v(\psi) = T.$

Definition 8 $\varphi \vDash_{\mathcal{F}_1} \psi =_{df} \forall v : v(\psi) = F \Rightarrow v(\varphi) = F.$

What kinds of systems can be employed to axiomatize the entailment relations determined by these definitions? Answering this question is not as easy as it might seem.

One can try to reorganize systems $\vdash_H$ and $\vdash_{dH}$ so that only rules of the FMLA-FMLA type are left intact. Consider $\vdash_H$. The only rule of this system that is responsible for the multiplication of premises is (R3). However, its direct single-premise counterpart in accordance with the relationship between SET-FMLA and FMLA-FMLA entailments established at the end of the previous section, turns out to be a trivial identity statement:

$$\frac{\varphi \wedge \psi}{\varphi \wedge \psi}.$$

This rule is easily derivable from the rest of the rules of $\vdash_H$, and is thus redundant. The situation with $\vdash_{dH}$ and its multiple-conclusions rule (R3_d) is analogous. Thus, a direct "singularization" of $\vdash_H$ and $\vdash_{dH}$ will not work (being incomplete), because the resulting systems will lack the principle of conjunction elimination and disjunction introduction respectively.

[9] The dual Belnap logic is studied in detail in [31]. The FMLA-SET entailment relation for this logic can be defined as follows:

Definition 6 $\varphi \vDash_{\mathcal{DB}} \Delta =_{df} \forall v : (\forall \psi \in \Delta : f \in v(\psi)) \Rightarrow f \in v(\varphi).$

This definition is explicitly based on $\{T, N\}$ as the set of designated truth values, implementing thus the second half of Belnap's regulation for a valid inference. System $\vdash_{dH}$ is sound and complete with respect to Definition 6.

Let us take a closer look at the entailment relations determined by Definitions 7 and 8. Both these relations are proper extensions of the relation defined by Definition 3, as the following lemma shows:

Lemma 1 *For any* $\varphi, \psi \in \mathcal{L}$:

1. $\varphi \vDash_{\mathcal{B}_1} \psi \Rightarrow \varphi \vDash_{\mathcal{T}_1} \psi$;
2. $\varphi \vDash_{\mathcal{B}_1} \psi \Rightarrow \varphi \vDash_{\mathcal{F}_1} \psi$.

Proof 1. Consider arbitrary $\varphi, \psi \in \mathcal{L}$, and let $\forall v : (t \in v(\varphi)) \Rightarrow t \in v(\psi)$. For every valuation v define its dual v^*, such that $t \in v^*(p) \Leftrightarrow f \notin v(p)$, and $f \in v^*(p) \Leftrightarrow t \notin v(p)$. A direct induction extends this valuation to any formula of the language. Moreover, it is not difficult to show that $v^{**}(\varphi) = v(\varphi)$, for any φ. Now, consider an arbitrary valuation v', such that $t \in v'(\varphi)$, and $f \notin v'(\varphi)$. First, we have $t \in v'(\psi)$. Assume $f \in v'(\psi)$. Then, by definition of dual valuation, $t \notin v'^*(\psi)$. Hence, by the lemma condition, $t \notin v'^*(\varphi)$. Thus, $f \in v'^{**}(\varphi)$, whence $f \in v'(\varphi)$, a contradiction.

Moreover, $\vDash_{\mathcal{T}_1}$ is a proper extension of $\vDash_{\mathcal{B}_1}$. In particular, $\sim\varphi \wedge (\varphi \vee \psi) \vDash_{\mathcal{T}_1} \psi$, but $\sim\varphi \wedge (\varphi \vee \psi) \nvDash_{\mathcal{B}_1} \psi$.

2. The proof is analogous. □

This lemma suggests a construction of the proof systems for the FMLA-FMLA exactly true and non-falsity logics as extensions of a suitably formulated consequence system for the first-degree entailment logic, in the way that Pietz and Rivieccio do with SET-FMLA Belnap's logic, by adding the rule γ (or disjunctive syllogism) to it. However, the analogous manipulations with $\mathbf{E}_{\mathrm{fde}}$ or $\mathbf{R}_{\mathrm{fde}}$ do not produce the desired effect, cf. [35, p. 1304].

Indeed, it is very well known that adding to $\mathbf{E}_{\mathrm{fde}}$ the principle of disjunctive syllogism

$$\sim\varphi \wedge (\varphi \vee \psi) \vdash \psi$$

as an axiom, leads to a system of classical consequence, whereas adding the above principle to $\mathbf{R}_{\mathrm{fde}}$ produces the first-degree fragment of Kleene's logic. Similarly, adding to $\mathbf{E}_{\mathrm{fde}}$ the principle of dual disjunctive syllogism

$$\varphi \vdash \sim\psi \vee (\psi \wedge \varphi)$$

as an axiom, again leads to a system of classical consequence, whereas adding the above principle to $\mathbf{R}_{\mathrm{fde}}$ produces the first-degree fragment of Priest's logic of paradox.[10]

[10]By using disjunctive syllogism (or Ackermann's rule γ) one can easily derive the principle *ex falso quodlibet*: $\varphi \wedge \sim\varphi \vdash \psi$, whereas by using dual disjunctive syllogism (dual γ) one can easily derive the principle *verum ex quodlibet*: $\varphi \vdash \psi \vee \sim\psi$. In $\mathbf{E}_{\mathrm{fde}}$ (unlike in $\mathbf{R}_{\mathrm{fde}}$) these principles are interderivable, due to contraposition. For further details consult, e.g., [12, p. 15].

In [30] the following two binary consequence systems are proposed for the exactly true logic and the non-falsity logic:

System $\mathbf{ETL}_1$

(1) $\varphi \wedge \psi \vdash \varphi$

(2) $\varphi \wedge \psi \vdash \psi$

(3) $\varphi \vdash \varphi \vee \psi$

(4) $\varphi \vee \psi \vdash \psi \vee \varphi$

(5) $\varphi \vee \varphi \vdash \varphi$

(6) $\varphi \vee (\psi \vee \chi) \vdash (\varphi \vee \psi) \vee \chi$

(7) $\varphi \vee (\psi \wedge \chi) \vdash (\varphi \vee \psi) \wedge (\varphi \vee \chi)$

(8) $(\varphi \vee \psi) \wedge (\varphi \vee \chi) \vdash \varphi \vee (\psi \wedge \chi)$

(9) $\varphi \vee \psi \vdash {\sim}{\sim}\varphi \vee \psi$

(10) ${\sim}{\sim}\varphi \vee \psi \vdash \varphi \vee \psi$

(11) ${\sim}(\varphi \vee \psi) \vee \chi \vdash ({\sim}\varphi \wedge {\sim}\psi) \vee \chi$

(12) $({\sim}\varphi \wedge {\sim}\psi) \vee \chi \vdash {\sim}(\varphi \vee \psi) \vee \chi$

(13) ${\sim}(\varphi \wedge \psi) \vee \chi \vdash ({\sim}\varphi \vee {\sim}\psi) \vee \chi$

(14) $({\sim}\varphi \vee {\sim}\psi) \vee \chi \vdash {\sim}(\varphi \wedge \psi) \vee \chi$

(15) ${\sim}\varphi \wedge (\varphi \vee \psi) \vdash \psi$

$(r1)$ $\varphi \vdash \psi$, $\psi \vdash \chi$ / $\varphi \vdash \chi$

$(r2)$ $\varphi \vdash \psi$, $\varphi \vdash \chi$ / $\varphi \vdash \psi \wedge \chi$

System $\mathbf{NFL}_1$

(1′) $\varphi \vdash \varphi \vee \psi$

(2′) $\psi \vdash \varphi \vee \psi$

(3′) $\varphi \wedge \psi \vdash \varphi$

(4′) $\varphi \wedge \psi \vdash \psi \wedge \varphi$

(5′) $\varphi \vdash \varphi \wedge \varphi$

(6′) $(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)$

(7′) $(\varphi \wedge \psi) \vee (\varphi \wedge \chi) \vdash \varphi \wedge (\psi \vee \chi)$

(8′) $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$

(9′) ${\sim}{\sim}\varphi \wedge \psi \vdash \varphi \wedge \psi$

(10′) $\varphi \wedge \psi \vdash {\sim}{\sim}\varphi \wedge \psi$

(11′) $({\sim}\varphi \vee {\sim}\psi) \wedge \chi \vdash {\sim}(\varphi \wedge \psi) \wedge \chi$

(12′) ${\sim}(\varphi \wedge \psi) \wedge \chi \vdash ({\sim}\varphi \vee {\sim}\psi) \wedge \chi$

(13′) $({\sim}\varphi \wedge {\sim}\psi) \wedge \chi \vdash {\sim}(\varphi \vee \psi) \wedge \chi$

(14′) ${\sim}(\varphi \vee \psi) \wedge \chi \vdash ({\sim}\varphi \wedge {\sim}\psi) \wedge \chi$

(15′) $\varphi \vdash {\sim}\psi \vee (\psi \wedge \varphi)$

$(r1)$ $\varphi \vdash \psi$, $\psi \vdash \chi$ / $\varphi \vdash \chi$

$(r3)$ $\varphi \vdash \chi$, $\psi \vdash \chi$ / $\varphi \vee \psi \vdash \chi$

These systems result from the certain binary restructuring systems $\vdash_H$ and $\vdash_{dH}$, mainly by means of introducing the indirect inference rules for binary consequence expressions, instead of the corresponding two-premises(-conclusions) direct rules.

It is easy to see that these systems are sound with respect to Definitions 7 and 8 respectively. Moreover, in [30] it is shown that the systems determined by (1)–(14) + $(r1)$, $(r2)$ and (1′)–(14′) + $(r1)$, $(r3)$ are deductively equivalent to each other, as well as to both $\mathbf{E}_{\text{fde}}$ and $\mathbf{R}_{\text{fde}}$. It means that both $\mathbf{ETL}_1$ and $\mathbf{NFL}_1$ are indeed the proper extensions of the first-degree entailment logic.

However, the problem of establishing completeness for these systems was left open in [30]. In what follows we will address this problem and elaborate on a certain new method of its resolving.

3 A Bi-consequence System for Exactly True Logic

It is worth observing that the logic of first-degree entailment admits a very elegant completeness proof by a canonical model construction in terms of prime theories. The properties of logical connectives secured by $\mathbf{E}_{\text{fde}}$ (and any proof system deductively equivalent to it) not only allow for a very natural definition of a canonical valuation through a membership of a propositional variable or its negation in a prime theory, but ensure also a straightforward extension of such canonical valuation to any compound formula, enabling thus the completion of the canonical model for the whole language. Dunn in [11, pp. 40–42] presents "a 'sanitized' version of that proof that does not rely on a reading knowledge of 'algebraese'." This proof is directly applicable to Belnap's logic in any logical framework. This is a Henkin-style "pure logical" completeness proof, and we believe that it is not an exaggeration to say that possessing such a proof is an evidence of perfection for a proof system.

By contrast, the situation with exactly true logic is not so perfect. Certain specific features of $\models_{\mathcal{T}_1}$ make the construction of a suitable canonical model problematic. One such feature, characterized by Pietz and Rivieccio as "most unusual" [25, p. 129], is that $(\varphi \wedge {\sim}\varphi) \vee (\psi \wedge {\sim}\psi) \models_{\mathcal{T}_1} \chi$ is not valid even though both $\varphi \wedge {\sim}\varphi \models_{\mathcal{T}_1} \chi$ and $\psi \wedge {\sim}\psi \models_{\mathcal{T}_1} \chi$ are.[11] To check this, take $v(\varphi) = B$, $v(\psi) = N$ and $v(\chi) = F$. This property, called "anti-primeness" by Pietz and Rivieccio, indicates the fact that the principle of *disjunction elimination* is not admissible in **ETL**. This fact greatly complicates finding a suitable canonical valuation for the exactly true logic, extendable to disjunctive formulas. Moreover, Pietz and Rivieccio presume, that failure of the disjunction elimination (as a general principle) "will not make it easy to find a nice sequent calculus for this logic" [25, p. 129].[12]

Thus, Pietz and Rivieccio provide a completeness proof for **ETL** by using methods from abstract algebraic logic, most crucially, by formulating a reduced matrix for **ETL**, and employing the well-known fact that any logic is complete with respect to the class of its reduced matrix models, see [25, p. 131–133]. Still, following Dunn, one might wish to see a "purely logical" (Henkin-style) completeness proof for the exactly true logic (and its dual) as well.

To approach this task, we will effectively make use of a specific proof-theoretic formalism falling under the category of a *bi-consequence system*. Generally systems of this kind enable simultaneous manipulations with *two* consequence relations within *one* deductive framework. The mainstream of logical development is to consider logical systems as codifying valid consequence relations—one relation for each such system. However, as observed in [33, p. 144]: "characterization of logic as the theory of valid inferences ... does not preclude that there may be more than just one kind of valid inferences".

Indeed, the idea of a "peaceful co-existence of two entailment and two deducibility relations in one and the same logic" [33, p. 144] has been advanced in the literature by some authors. For example, Dimiter Vakarelov in [36, p. 206] introduces the notion of a *bi-consequence system* "as an abstract system in the form $(Sen, \vdash, \succ)$, where Sen is a nonempty set, whose elements are called sentences and $\vdash$ and $\succ$ are two relations between finite set of sentences, called respectively strong and weak consequence relations and satisfying some axioms like structural rules in Gentzen

[11] Observe that the exactly true logic is not paraconsistent, because Definition 7 validates the principle of *explosion*: $\varphi \wedge {\sim}\varphi \models_{\mathcal{T}_1} \psi$ (by no valuation a contradiction takes the value T). Still it is paracomplete, since $\varphi \models_{\mathcal{T}_1} \psi \vee {\sim}\psi$ is not valid (let φ take the value T, and ψ value B or N).

[12] However, see a formulation of a "four-sided sequents calculus" by Stefan Wintein and Reinhard Muskens in [39]. Their proof system belongs to the SET-FMLA framework, and provides another proof-theoretic formalism for grasping the entailment relation determined by Definition 4.

systems". This formalism is used for representing information systems of a certain kind, which turn out to be dual to some knowledge representation systems.

Stéphane Demri and Ewa Orłowska in [8] also consider bi-consequence systems as part of the analysis of various structures dealing with incomplete information. They observe a close relationship between information systems and bi-consequence systems, which is similar to the relationship between so-called "property systems" and (unary) consequence systems.

A bi-consequence approach to deductive frameworks has also been put forward in [33] in the context of some far-reaching generalizations of the four-valued logics by the notion of a *trilattice*. In particular, the trilattice of generalized truth values $SIXTEEN_3$ investigated in [33] is equipped with two "logical orderings"—a truth order and a falsity order—that separately determine the properties of logical connectives as well as the corresponding entailment relations. The unified logic of $SIXTEEN_3$ has been conceived as a bi-consequence system comprising two kinds of deducibility relations: $\vdash_t$ and $\vdash_f$, for "truth-consequence" and "falsity-consequence", respectively. A complete axiomatization of this logic has been provided in [23] in the form of a *bi-calculus*.

This approach has been further generalized in [29] by the notion of a *multi-consequence system*, where any such system can be furnished with as many consequence relations as is needed for the purposes of a logical analysis. The multi-consequence logics (and corresponding truth value multilattices) that may comprise several entailment relations are motivated there by an observation that "by a logical reasoning we can be interested not only in informational content, truth content, or falsity content, but also in some other possible characterizations of the given truth-values, such as constructivity, cf. [32], (un)certainty, cf. [40], modality, cf. [27], or other kinds of 'adverbial qualifications', cf. [20], by which truth values can naturally be ordered" [29, p. 205].

In what follows we will develop a specific version of a bi-consequence formalism, which can be described as a *two-level bi-consequence system*. Two consequence relations comprised by such a framework are related subordinately, so that one of these relations (on level 1) is treated as the basic for the resulting logic, and the other is introduced (on level 2) as an extension of this basic relation. As a result one obtains an exact deductive characterization of the entailment relation that corresponds to the second-level consequence.

In the case of the FMLA-FMLA exactly true logic the characteristic deductive principles of this logic will be directly built over the basic system of the first-degree entailment, which is coincident with $\mathbf{E}_{\mathrm{fde}}$. The resulting bi-consequence system allows the desired Henkin-style completeness proof with respect to Definition 7. By showing that this system is deductively equivalent to $\mathbf{ETL}_1$ the completeness of the latter will be secured as well.

Thus, the bi-consequence system $\mathbf{ETL}_1^2$ is determined by the following axiom schemata and rules of inference:

$(a1)$ $\varphi \wedge \psi \vdash_1 \varphi$
$(a2)$ $\varphi \wedge \psi \vdash_1 \psi$
$(a3)$ $\varphi \vdash_1 \varphi \vee \psi$

(*a*4) $\psi \vdash_1 \varphi \vee \psi$
(*a*5) $\varphi \wedge (\psi \vee \chi) \vdash_1 (\varphi \wedge \psi) \vee \chi$
(*a*6) $\varphi \vdash_1 {\sim}{\sim}\varphi$
(*a*7) ${\sim}{\sim}\varphi \vdash_1 \varphi$
(*a*8) $\varphi \wedge ({\sim}\varphi \vee \psi) \vdash_2 \psi$

(*r*1) $\varphi \vdash_1 \psi,\ \psi \vdash_1 \chi\ /\ \varphi \vdash_1 \chi$
(*r*2) $\varphi \vdash_1 \psi,\ \varphi \vdash_1 \chi\ /\ \varphi \vdash_1 \psi \wedge \chi$
(*r*3) $\varphi \vdash_1 \chi,\ \psi \vdash_1 \chi\ /\ \varphi \vee \psi \vdash_1 \chi$
(*r*4) $\varphi \vdash_1 \psi\ /\ {\sim}\psi \vdash_1 {\sim}\varphi$
(*r*5) $\varphi \vdash_2 \psi,\ \psi \vdash_2 \chi\ /\ \varphi \vdash_2 \chi$
(*r*6) $\varphi \vdash_2 \psi,\ \varphi \vdash_2 \chi\ /\ \varphi \vdash_2 \psi \wedge \chi$
(*r*7) $\varphi \vdash_1 \psi\ /\ \varphi \vdash_2 \psi$

Some remarks on $\mathbf{ETL}_1^2$ are given as follows.

1. $\mathbf{ETL}_1^2$ is generally defined as a triple $\langle \mathcal{L}, \vdash_1, \vdash_2 \rangle$, where $\vdash_1$ and $\vdash_2$ are distinct consequence relations, each characterized by the corresponding axioms and rules of inference.
2. In label $\mathbf{ETL}_1^2$ the superscript stands for the number of consequence relations used in a formulation of a system, and the subscript marks the singularity restriction on the consequence expressions involved.
3. The main task of $\mathbf{ETL}_1^2$ is an axiomatization of the relation $\vdash_2$, i.e., this system is designed to obtain the set of all valid consequences of the form $\varphi \vdash_2 \psi$. The role of $\vdash_1$ is basically a supporting one. An *inference (proof)* in $\mathbf{ETL}_1^2$ of the consequence $\varphi \vdash_2 \psi$ is a finite list of consequence expressions, where every list item is either an axiom or results from preceding elements of the list by an inference rule application, and the last item in the list is $\varphi \vdash_2 \psi$. If there is an inference of $\varphi \vdash_2 \psi$ in $\mathbf{ETL}_1^2$, then we say that $\varphi \vdash_2 \psi$ is *provable* in $\mathbf{ETL}_1^2$ (or ETL-provable), labeling it with $\varphi \vdash_{\mathrm{etl}} \psi$.
4. The $\vdash_1$-part of $\mathbf{ETL}_1^2$ determined by (*a*1)–(*a*7), (*r*1)–(*r*4) is exactly $\mathbf{E}_{\mathrm{fde}}$. To highlight this point we preserve the same numbering for the analogous axioms and rules in both systems. If there is a proof of a consequence $\varphi \vdash_1 \psi$ purely within the $\vdash_1$-part of $\mathbf{ETL}_1^2$, we say that it is FDE-provable and mark it with $\varphi \vdash_{\mathrm{fde}} \psi$.
5. The properties of disjunction elimination and contraposition do not hold for $\vdash_2$ in full generality, since $\vdash_2$-versions of (*r*3) and (*r*4) are *not admissible*. However, restricted versions of these rules hold, as it is not difficult to see: $\varphi \vdash_1 \chi,\ \psi \vdash_1 \chi\ /\ \varphi \vee \psi \vdash_2 \chi;\ \varphi \vdash_1 \psi\ /\ {\sim}\psi \vdash_2 {\sim}\varphi$.
6. Generally speaking, $\varphi \vdash_{\mathrm{fde}} \psi \Rightarrow \varphi \vdash_{\mathrm{etl}} \psi$. Moreover, the set of valid $\vdash_2$-consequences is a proper extension of the set of valid $\vdash_1$-consequences, due to axiom (*a*8) and rule (*r*7). The latter rule reflects a hierarchic interconnection between the consequences, revealing $\vdash_1$ as a relation of the first level, and $\vdash_2$ as a relation of the second level.

For the sake of illustration consider inferences in $\mathbf{ETL}_1^2$ of the consequences $\varphi \wedge {\sim}\varphi \vdash_2 \psi$ and $\varphi \vee ({\sim}\psi \vee {\sim}\chi) \vdash_2 \varphi \vee {\sim}(\psi \wedge \chi)$.

1.	$\varphi \wedge \sim\varphi \vdash_1 \varphi$	$(a1)$
2.	$\varphi \wedge \sim\varphi \vdash_1 \sim\varphi$	$(a2)$
3.	$\sim\varphi \vdash_1 \sim\varphi \vee \psi$	$(a3)$
4.	$\varphi \wedge \sim\varphi \vdash_1 \sim\varphi \vee \psi$	2, 3: $(r1)$
5.	$\varphi \wedge \sim\varphi \vdash_1 \varphi \wedge (\sim\varphi \vee \psi)$	1, 4: $(r3)$
6.	$\varphi \wedge \sim\varphi \vdash_2 \varphi \wedge (\sim\varphi \vee \psi)$	5: $(r7)$
7.	$\varphi \wedge (\sim\varphi \vee \psi) \vdash_2 \psi$	$(a8)$
8.	$\varphi \wedge \sim\varphi \vdash_2 \psi$ 6, 7:	$(r5)$

1.	$\psi \wedge \chi \vdash_1 \psi$	$(a1)$
2.	$\psi \wedge \varphi \vdash_1 \chi$	$(a2)$
3.	$\sim\psi \vdash_1 \sim(\psi \wedge \chi)$	1: $(r4)$
4.	$\sim\psi \vdash_1 \sim(\psi \wedge \chi)$	2: $(r4)$
5.	$\sim\psi \vee \sim\chi \vdash_1 \sim(\psi \wedge \chi)$	3, 4: $(r3)$
6.	$\varphi \vdash_1 \varphi \vee \sim(\psi \wedge \chi)$	$(a3)$
7.	$\sim(\psi \wedge \chi) \vdash_1 \varphi \vee \sim(\psi \wedge \chi)$	$(a4)$
8.	$\sim\psi \vee \sim\chi \vdash_1 \varphi \vee \sim(\psi \wedge \chi)$	5, 7: $(r1)$
9.	$\varphi \vee (\sim\psi \vee \sim\chi) \vdash_1 \varphi \vee \sim(\psi \wedge \chi)$	6, 8: $(r3)$
10.	$\varphi \vee (\sim\psi \vee \sim\chi) \vdash_2 \varphi \vee \sim(\psi \wedge \chi)$	9: $(r7)$

System $\mathbf{ETL}_1^2$ is *sound* with respect to Definition 7.

Theorem 1 *For any $\varphi, \psi \in \mathcal{L} : \varphi \vdash_{etl} \psi \Rightarrow \varphi \models_{\mathcal{T}_1} \psi$.*

Proof A direct check shows that all axioms of $\mathbf{ETL}_1^2$ are valid with respect to Definition 7. It is also not difficult to see that for every inference rule $(r1)$–$(r6)$ the conclusion of the rule is valid with respect to Definition 7, provided the premises are. For the preservation of ETL-validity of $(r7)$ see Lemma 1. □

For a completeness proof we employ a canonical model construction in terms of FDE-theories. Let a *theory* $\mathcal{T}$ be a set of formulas closed under the provable $\vdash_1$ and under $\wedge$. That is, for any $\varphi, \psi \in \mathcal{L}$:

$$\varphi \in \mathcal{T}, \varphi \vdash_{\text{fde}} \psi \Rightarrow \psi \in \mathcal{T}; \tag{1}$$

$$\varphi \in \mathcal{T}, \psi \in \mathcal{T} \Rightarrow \varphi \wedge \psi \in \mathcal{T}. \tag{2}$$

As usual, a theory $\mathcal{T}$ is *prime* if the following holds:

$$\varphi \vee \psi \in \mathcal{T} \Rightarrow \varphi \in \mathcal{T} \text{ or } \psi \in \mathcal{T}. \tag{3}$$

For any theory $\mathcal{T}$, and any formula φ, let $\mathcal{T} + \varphi$ stands for the smallest theory $\mathcal{T}'$, such that $\mathcal{T} \cup \varphi \subseteq \mathcal{T}'$. We will need the following version of Lindenbaum lemma:

Lemma 2 *Let $\varphi \nvdash_{etl} \psi$. Then there is a prime theory $\mathcal{T}$, such that $\varphi \in \mathcal{T}$, $\sim\varphi \notin \mathcal{T}$, and $\psi \notin \mathcal{T}$.*

Proof This is a suitable adjustment of the proof of Lemma 8 from [12]. Enumerate all the sentences of $\mathcal{L}$: $\varphi_1, \varphi_2, \ldots$, and then built up a series of theories starting with $\mathcal{T}_0 = \{\psi' : \varphi \vdash_{\text{fde}} \psi'\}$. Clearly, $\varphi \in \mathcal{T}_0$. By the lemma condition and $(r7)$, $\varphi \nvdash_{\text{fde}} \psi$, and hence, $\psi \notin \mathcal{T}_0$. Moreover, $\sim\varphi \notin \mathcal{T}_0$. Indeed, assume $\sim\varphi \in \mathcal{T}_0$. Then $\varphi \vdash_{\text{fde}} \sim\varphi$.

By $(r7)$ $\varphi \vdash_{\text{etl}} {\sim}\varphi$. Since $\varphi \vdash_{\text{etl}} \varphi$, by $(r6)$ we have $\varphi \vdash_{\text{etl}} \varphi \wedge {\sim}\varphi$. From this and $\varphi \wedge {\sim}\varphi \vdash_{\text{etl}} \psi$ we obtain by $(r5)$ $\varphi \vdash_{\text{etl}} \psi$, contrary to the lemma condition.

For every $\mathcal{T}_n$ define $\mathcal{T}_{n+1}$ as follows: (1) if $\mathcal{T}_n + \varphi_n \vdash_{\text{etl}} {\sim}\varphi \vee \psi$, then $\mathcal{T}_{n+1} = \mathcal{T}_n$; (2) if $\mathcal{T}_n + \varphi_n \nvdash_{\text{etl}} {\sim}\varphi \vee \psi$, then $\mathcal{T}_{n+1} = \mathcal{T}_n + \varphi_n$.

The required theory $\mathcal{T}$ can be defined as the union of all the $\mathcal{T}_n$. It is easy to see that $\mathcal{T}$ is the maximal theory containing φ with respect to the property of not having ${\sim}\varphi \vee \psi$. To show that $\mathcal{T}$ is also prime, assume that there are $\chi, \chi' \notin \mathcal{T}$, such that $\chi \vee \chi' \in \mathcal{T}$. Consider two theories $\mathcal{T} + \chi$ and $\mathcal{T} + \chi'$. By the above maximality both these theories must contain ${\sim}\varphi \vee \psi$. Hence, there is a conjunction of members of $\mathcal{T}$, say τ, such that $\tau \wedge \chi \vdash_{\text{fde}} {\sim}\varphi \vee \psi$, and $\tau \wedge \chi' \vdash_{\text{fde}} {\sim}\varphi \vee \psi$. By $(r3)$ we get $(\tau \wedge \chi) \vee (\tau \wedge \chi') \vdash_{\text{fde}} {\sim}\varphi \vee \psi$. By using (a5), we obtain $\tau \wedge (\chi \vee \chi') \vdash_{\text{fde}} {\sim}\varphi \vee \psi$. Hence, ${\sim}\varphi \vee \psi \in \mathcal{T}$, a contradiction.

To finish the proof we have to show that ${\sim}\varphi \notin \mathcal{T}$ and $\psi \notin \mathcal{T}$, which is an easy task by using $(a3)$ and $(a4)$. □

The following valuation lemma is quite standard, given that we deal with theories defined exclusively with respect to FDE-consequence $\vdash_1$.

Lemma 3 *Let $\mathcal{T}$ be a prime theory, and define a canonical valuation v_τ so that $t \in v_\tau(p)$ iff $p \in \mathcal{T}$, and $f \in v_\tau(p)$ iff ${\sim}p \in \mathcal{T}$. Then Definition 1 holds for the canonical valuation so defined.*

Proof See, e.g., proof of Lemma 10 in [12]. □

We are now in a position to establish completeness of $\mathbf{ETL}_1^2$ with respect to Definition 7.

Theorem 2 *For any $\varphi, \psi \in \mathcal{L} : \varphi \vDash_{\mathcal{T}_1} \psi \Rightarrow \varphi \vdash_{etl} \psi$.*

Proof Suppose, $\varphi \nvdash_{\text{etl}} \psi$. By Lemma 2, there is a prime theory $\mathcal{T}$, such that $\varphi \in \mathcal{T}$, ${\sim}\varphi \notin \mathcal{T}$, and $\psi \notin \mathcal{T}$. Consider canonical valuation defined in Lemma 3. We have $t \in v_\tau(\varphi)$, $f \notin v_\tau(\varphi)$, and $t \notin v_\tau(\psi)$. That is, $v_\tau(\varphi) = T$, and $v_\tau(\psi) \neq T$. Hence, $\varphi \nvDash_{\mathcal{T}_1} \psi$. □

It remains to demonstrate the deductive equivalence of $\mathbf{ETL}_1$ and $\mathbf{ETL}_1^2$.

Theorem 3 *For any $\varphi, \psi \in \mathcal{L} : \varphi \vdash \psi$ is provable in* $\mathbf{ETL}_1$ *if and only if $\varphi \vdash_2 \psi$ is provable in* $\mathbf{ETL}_1^2$.

Proof $\Rightarrow$: It is not difficult to see that the $\vdash_2$-version of every axiom of $\mathbf{ETL}_1$ is derivable in $\mathbf{ETL}_1^2$, and the inference rules of $\mathbf{ETL}_1$ are also inference rules with respect to $\vdash_2$-derivability.

$\Leftarrow$: Any axiom of $\mathbf{ETL}_1^2$ is derivable in $\mathbf{ETL}_1$. As an example, show a derivation in $\mathbf{ETL}_1$ for $(a6)$:

1. $\varphi \vee {\sim\sim}\varphi \vdash {\sim\sim}\varphi \vee {\sim\sim}\varphi$ (9)
2. $\varphi \vdash \varphi \vee {\sim\sim}\varphi$ (3)
3. ${\sim\sim}\varphi \vee {\sim\sim}\varphi \vdash {\sim\sim}\varphi$ (5)

4. $\varphi \vdash \sim\sim\varphi$ 1, 2, 3: $(r1)$, twice

Now, let $\varphi \vdash_2 \psi$ is obtained by an inference rule of $\mathbf{ETL}_1^2$. It can be $(r5)$, $(r6)$ or $(r7)$. The $\vdash$-versions of the first two are also inference rules of $\mathbf{ETL}_1$. The case with $(r7)$ trivially holds by inductive hypothesis. □

Theorem 4 $\mathbf{ETL}_1$ *is sound and complete with respect to Definition 7.*

Proof Theorem 1, Theorem 2, and Theorem 3. □

There is an interesting question concerning a possible philosophical understanding of the proposed bi-consequence formalism. Intuitively, the construction of proof systems with several consequence relations may be justified by special requirements for a "computer based logic".[13] Whereas a human being may find it difficult to operate simultaneously with several entailment relations, it seems much more natural for an "electronic brain", especially when performing parallel computations. Even though the system $\mathbf{ETL}_1^2$ has emerged from the need to provide a completeness proof for the corresponding standard binary consequences system, it may well have value in itself. Moreover, a comparison of two kinds of systems reveals certain advantages of the bi-consequence construction. While $\mathbf{ETL}_1$ may look rather cumbersome and not very user-friendly, its bi-consequence counterpart $\mathbf{ETL}_1^2$ is clearly more compact and more operational in its deductive applications. Within certain boundaries, which are precisely defined by the relation $\vdash_1$, it can draw upon the entire deductive power of the first-degree entailment (including contraposition), thus resulting in a considerable simplification of derivations.

This construction is analogous to a technique known from certain formalizations of the non-monotonic reasoning, where a non-monotonic consequence $\mid\sim$ is explicated as a *supraclassical* relation, i.e. as an "ampliative extension" of the classical consequence $\models$ subject to the following stipulation: "if $\mid\sim$ is supraclassical, then $\varphi \models \psi$ is a sufficient condition for $\varphi \mid\sim \psi$. Thus supraclassical consequence relations build on the pairs in $\models$ by adding new inference-pairs legitimised by the agent's heuristic information" [19, p. 327].[14]

Similarly, in the case of the exactly true logic some additional to the first-degree entailment of relevant logic inference-pairs emerge, legitimized (in this case) by a new approach to designated truth-values. Thus, in $\mathbf{ETL}_1^2$, as well as in other bi-consequence systems based on $\mathbf{E}_{\mathrm{fde}}$, $\vdash_2$ in fact represents a *suprarelevant* consequence relation built on the first-degree relevant consequence $\vdash_1$. Remarkably, the rule $(r7)$, which essentially expresses this idea of a suprarelevancy, allows for a fine tuning of a wide variety of "suprarelevant binary systems" by picking out specific

[13] It may be illuminating in this respect to recall Belnap's initial motivation for constructing a "useful four-valued logic" of "how a computer should think".

[14] Incidentally, we find in [19] an interesting quote from Johan van Benthem in defense of the plurality of consequences: "The idea that logic is about just one notion of 'logical consequence' is actually one very particular historical stance. It was absent in the work of the great pioneer Bernard Bolzano, who thought that logic should chart the many different consequence relations that we have, depending on the reasoning task at hand" [37, p. 72].

axioms and rules for $\vdash_2$ without collapsing it into classical consequence relation. In Section 5 we will outline a family of such bi-consequence systems based on the logic of first-degree entailment.

4 A Bi-consequence System for Non-falsity Logic

As already observed, the entailment relation of the non-falsity logic $\vDash_{\mathcal{F}_1}$ (Definition 7) is dual to the entailment relation of the exactly true logic $\vDash_{\mathcal{T}_1}$. This duality finds its exact formulation in the following lemma:

Lemma 4 *Let φ, ψ be any sentences of $\mathcal{L}$, and let φ^d be obtained from φ by interchanging between $\wedge$ and $\vee$, and replacing every atomic sentence with its negation (and likewise for ψ^d). Then $\varphi \vDash_{\mathcal{T}_1} \psi \Leftrightarrow \psi^d \vDash_{\mathcal{F}_1} \varphi^d$.*

Proof An easy induction on the length of a formula gives for any formula φ, and for any valuation v: $t \in v(\varphi) \Leftrightarrow f \in v(\varphi^d)$, and $f \in v(\varphi) \Leftrightarrow t \in v(\varphi^d)$. The main claim of the lemma follows from this fact. □

Note, that $(a8)$ and $(r6)$ are the only axiom and rule of $\mathbf{ETL}_1^2$, which are not ETL-dualizable, i.e. the dual versions of $(a8)$ and $(r6)$ are not ETL-valid. However, in view of Lemma 4, dualizations of $(a8)$ and $(r6)$ turn out to be valid principles of non-falsity logic. Thus, a bi-consequence system for the non-falsity FMLA-FMLA logic $\mathbf{NFL}_1^2$ can be obtained from $\mathbf{ETL}_1^2$ by a simple substitution of $(a8)$ and $(r6)$ for their dual versions:

$(a8)^d$ $\quad \psi \vdash_2 \varphi \vee ({\sim}\varphi \wedge \psi)$
$(r6)^d$ $\quad \varphi \vdash_2 \chi\ ,\ \psi \vdash_2 \chi \ /\ \varphi \vee \psi \vdash_2 \chi$

System $\mathbf{NFL}_1^2$ is also defined as a triple $\langle \mathcal{L}, \vdash_1, \vdash_2 \rangle$, where $\vdash_2$ is built over $\vdash_1$ as its proper extension. Consequence $\varphi \vdash_2 \psi$ is *provable* in $\mathbf{NFL}_1^2$ (NFL-provable) if and only if there exists a finite list of consequence expressions (either with $\vdash_1$ or with $\vdash_2$, ad lib), where every list item is either an axiom of $\mathbf{NFL}_1^2$ or results from the preceding elements of the list by an inference rule application, and the last item in the list is $\varphi \vdash_2 \psi$. Such a consequence is labeled by $\varphi \vdash_{\mathrm{nfl}} \psi$.

We invite an interested reader to dualize the proofs of Theorem 1 and Theorem 2 to obtain the following soundness and completeness result for $\mathbf{NFL}_1^2$ with respect to Definition 8:

Theorem 5 *For any $\varphi, \psi \in \mathcal{L}$: $\varphi \vDash_{\mathcal{F}_1} \psi \Leftrightarrow \varphi \vdash_{nfl} \psi$.*

The dual version of Theorem 3 also holds, and thus, systems $\mathbf{NFL}_1$ and $\mathbf{NFL}_1^2$ are deductively equivalent. Hence, $\mathbf{NFL}_1$ is sound and complete with respect to Definition 8.

5 A Family of Bi-consequence Systems

There is an interesting line of research concerning the diversity of possible systems around $\mathbf{ETL}_1^2$ and $\mathbf{NFL}_1^2$. As observed in Section 2, adding either $\varphi \wedge {\sim}\varphi \vdash \psi$ or $\varphi \vdash \psi \vee {\sim}\psi$ to $\mathbf{E}_{\text{fde}}$ collapses it into classical logic. The only non-trivial non-classical extension of this system seems to be the first-degree entailment fragment of the system "R-Mingle", obtained by adopting the axiom of *safety*:[15]

$$\varphi \wedge {\sim}\varphi \vdash_1 \psi \vee {\sim}\psi.$$

In contrast, if we take the system $\mathbf{R}_{\text{fde}}$, the corresponding additions produce two further non-trivial non-classical logics—the first-degree fragments of either Kleene's strong three-valued logic, or Priest's logic of paradox. These three possible extensions of the first-degree entailment logic are generally well studied.

As shown in [28] and [1], Font's "Hilbert-style formulation" of Belnap's logic $\vdash_H$ allows for more subtle distinctions. In particular, the rule γ turns out to be not derivable in the system $\vdash_H + \varphi \wedge {\sim}\varphi \vdash \psi$, see [28, p. 329], which makes it a distinctive system in its own right. Moreover, there is an infinite denumerable chain of systems between this latter system and Kleene's logic, the n-th element of which is obtained by accepting the following rule for any $n \geq 1$:

$$\frac{(\varphi_1 \wedge {\sim}\varphi_1) \vee \ldots \vee (\varphi_n \wedge {\sim}\varphi_n)}{\psi},$$

ending with the system, which is the union of all the elements of the chain.

The results of Sections 3 and 4 suggest a possibility of the corresponding extensions of systems $\mathbf{ETL}_1$ and $\mathbf{NFL}_1$. Moreover, construction of the two-level bi-consequence systems provides us with an effective tool for obtaining non-trivial extensions of the first-degree entailment as a direct add-in of the given basis. Namely, the distinction between two consequence relations makes it possible to accept additional principles merely for the second-level consequence without trivializing the background (first-level) consequence or collapsing it into classical logic.

Consider the following additional second-level consequence expressions:

(a9) $\varphi \wedge ({\sim}\varphi \vee \psi) \vdash_2 \psi \vee ({\sim}\psi \wedge \varphi)$
(a10) $\varphi \wedge {\sim}\varphi \vdash_2 \psi$
(a11) $\varphi \vdash_2 \psi \vee {\sim}\psi$

Schema (a9) combines Ackerman's γ and its dual, and can thus be called *united* γ; (a10) and (a11) represent the principles *ex falso quodlibet* and *verum ex quodlibet* correspondingly. Provided $\mathbf{ETL}_1^2$ and $\mathbf{NFL}_1^2$ are defined as in Sections 3 and 4, one can define more bi-consequence systems of the FMLA-FMLA type as follows:

$\mathbf{FDE}_1^2 =$ (a1)–(a7), (r1)–(r4), (r7).
$\mathbf{SM}_1^2 = \mathbf{FDE}_1^2 +$ (a9), (r5).

[15] See conjecture formulated in footnote 3 of [30].

$\mathbf{EFQ}^2_1 = \mathbf{FDE}^2_1 + (a10), (r5), (r6)$.
$\mathbf{VEQ}^2_1 = \mathbf{FDE}^2_1 + (a11), (r5), (r6)^d$.
$\mathbf{K}^2_1 = \mathbf{FDE}^2_1 + (a8), (r5), (r6), (r6)^d$.
$\mathbf{LP}^2_1 = \mathbf{FDE}^2_1 + (a8)^d, (r5), (r6), (r6)^d$.
$\mathbf{RM}^2_1 = \mathbf{FDE}^2_1 + (a9), (r5), (r6), (r6)^d$.
$\mathbf{C}^2_1 = \mathbf{FDE}^2_1 + (a8), (a8)^d, (r5), (r6), (r6)^d$.

$\mathbf{FDE}^2_1$ is just a formulation of the first-degree entailment system closed under $\vdash_2$. It has only one rule for the second-level consequence relation, which ensures the above-mentioned closure, and no specific axioms for $\vdash_2$. As the next step we add the "second-level transitivity", and also accept the axiom of united γ. Safety becomes then derivable, as the following schema of inference shows:

1.	$(\varphi \wedge \sim\varphi) \vee (\varphi \wedge \psi) \vdash_1 \varphi \wedge (\sim\varphi \vee \psi)$	(distributivity)
2.	$\psi \vee (\sim\psi \wedge \varphi) \vdash_1 (\psi \vee \sim\psi) \wedge (\psi \vee \varphi)$	(distributivity)
3.	$(\psi \vee \sim\psi) \wedge (\psi \vee \varphi) \vdash_1 (\psi \vee \sim\psi)$	$(a1)$
4.	$(\varphi \wedge \sim\varphi) \vdash_1 (\varphi \wedge \sim\varphi) \vee (\varphi \wedge \psi)$	$(a3)$
5.	$\psi \vee (\sim\psi \wedge \varphi) \vdash_1 (\psi \vee \sim\psi)$	2, 3: $(r1)$
6.	$(\varphi \wedge \sim\varphi) \vdash_1 \varphi \wedge (\sim\varphi \vee \psi)$	4, 1: $(r1)$
7.	$\psi \vee (\sim\psi \wedge \varphi) \vdash_2 (\psi \vee \sim\psi)$	5: $(r7)$
8.	$(\varphi \wedge \sim\varphi) \vdash_2 \varphi \wedge (\sim\varphi \vee \psi)$	6: $(r7)$
9.	$\varphi \wedge (\sim\varphi \vee \psi) \vdash_2 \psi \vee (\sim\psi \wedge \varphi)$	$(a9)$
10.	$\varphi \wedge \sim\varphi \vdash_2 \psi \vee \sim\psi$	7, 8, 9: $(r5)$, twice

As mentioned above, safety is the characteristic principle of the first degree entailment fragment of the logic R-Mingle. However, the system so defined which we label by $\mathbf{SM}^2_1$ (for "sub-mingle"), is *not* a bi-consequence formulation of the first degree fragment of **RM**. The later system validates the following consequence expressions, which are not derivable in $\mathbf{SM}^2_1$ in the absence of conjunction introduction and disjunction elimination:

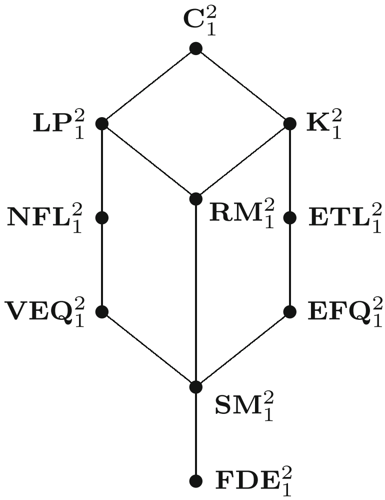
$$(\varphi \wedge \sim\varphi) \vee (\psi \wedge \sim\psi) \vdash_2 (\chi \vee \sim\chi) \tag{4}$$

$$(\varphi \wedge \sim\varphi) \vdash_2 (\psi \vee \sim\psi) \wedge (\chi \vee \sim\chi) \tag{5}$$

$$(\varphi \wedge \sim\varphi) \vee (\psi \wedge \sim\psi) \vdash_2 (\chi \vee \sim\chi) \wedge (\xi \vee \sim\xi) \tag{6}$$

We then proceed with adopting further rules and axioms for $\vdash_2$, thus making the second-level consequence richer and richer. System $\mathbf{EFQ}^2_1$ supplies $\vdash_2$ with the rule for conjunction introduction, but lacks disjunction elimination. $\mathbf{ETL}^2_1$ possesses disjunction elimination on the $\vdash_2$-level in a "polluted" form of an axiom which also involves negation. If we restore the disjunction elimination in full generality as an indirect rule of inference $(r6)^d$, we obtain the first-degree fragment of Kleene's three-valued (strong) logic $\mathbf{K}^2_1$. Dually we can move through systems $\mathbf{VEQ}^2_1$ and $\mathbf{NFL}^2_1$, up to the first-degree fragment of Priest's logic of paradox $\mathbf{LP}^2_1$. By uniting $\mathbf{K}^2_1$ and $\mathbf{LP}^2_1$ we arrive at the bi-consequence system of classical logic $\mathbf{C}^2_1$. There is also the third way to classical logic, which runs through the bi-consequence formulation of R-Mingle $\mathbf{RM}^2_1$.

Fig. 1 The kite of bi-consequence systems



The relations between these systems can be described by a "kite of bi-consequence systems" as represented in Fig. 1, where the bottom-up lines stand for the proper inclusion between the sets of valid consequences of the corresponding systems.

Note, that we can also consider some other intermediate systems that lie between the ones represented in Fig. 1. For example, one can consider a system obtained by adding to $\mathbf{FDE}_1^2$ axiom ($a10$) without the rule ($r6$), which will be situated between $\mathbf{SM}_1^2$ and $\mathbf{EFQ}_1^2$, as well as the dual system between $\mathbf{SM}_1^2$ and $\mathbf{VEQ}_1^2$. Likewise we can obtain a further system between $\mathbf{SM}_1^2$ and $\mathbf{EFQ}_1^2$ by merely changing ($a10$) to ($a8$). Moreover, by employing the results from [1] and [28], one can show that there is in fact infinitely many logical systems between $\mathbf{EFQ}_1^2$ and $\mathbf{K}_1^2$ (dually—between $\mathbf{EFQ}_1^2$ and $\mathbf{LP}_1^2$), each being definable by adding a version of ($a10$) (dually—($a11$)) for some n.

The diversity of bi-consequence systems so defined merits separate consideration. In particular, the issue of semantic characterization of systems between $\mathbf{SM}_1^2$ and $\mathbf{ETL}_1^2$, as well as $\mathbf{SM}_1^2$ and $\mathbf{NFL}_1^2$ is of special interest. We here only mention one definition, which introduces an entailment relation with a property of the forward preservation of T alongside the backward preservation of F:[16]

Definition 9 $\varphi \vDash_{\mathcal{TF}_1} \psi =_{df} \forall v : v(\varphi) = T \Rightarrow v(\psi) = T$ and $v(\psi) = F \Rightarrow v(\varphi) = F$.

It is not difficult to see that this definition validates ($a9$), but neither of ($a10$), ($a11$), (4)–(6), which suggests its faithfulness to $\mathbf{SM}_1^2$.

[16] As Andreas Kapsner informed the first author (personal communication), this definition was suggested by David Makinson in conversation.

6 Concluding Remarks and Future Work

In this paper we presented a solution to the problem of a deductive formalization of the FMLA-FMLA versions of Pietz and Rivieccio's exactly true logic and the non-falsity logic dual to it. The corresponding binary consequence systems proposed in [30] are shown to be sound and complete with respect to the entailment relations between single propositions based on Belnapian *true only* (T) and *false only* (F) as the designated and non-designated truth-value, respectively. The completeness of these systems is proved indirectly by means of a specific proof-theoretic construction, which (1) is deductively equivalent to the initial systems, and (2) can be equipped with the appropriate Henkin-style canonical models.

The resulting proof-theoretic formalism is constructed in the form of a two-level bi-consequence system, comprising two consequence relations hierarchically organized: the first relation is basic for the corresponding system, whereas the second one extends the basic relation. The set of consequences extended in this way is sound and complete with respect to the corresponding entailment relation to be deductively formalized.

It can generally be concluded that the binary consequence systems of the FMLA-FMLA type are quite useful in both "purely logical" and algebraic respects. Furthermore, the proposed bi-consequence construction not only presents a helpful and highly efficient technical tool for investigating the corresponding (standard) "mono-consequence" systems, but is also of interest in its own right.

From a logical point of view the binary consequence systems constitute the first-degree entailment fragments of the corresponding logics, that can further be formulated in the "full-fledged" languages with unrestricted implicational connectives. A possible direction for future research consists then in elaborating such logical systems dealing with a conditional that preserves the "exact truth" and its dual.[17] Moreover, the bi-consequence formalism opens up a possibility of developing logical systems simultaneously comprising *two* implicational connectives, one of which is the implication of system **E** or **R**, and the other is its augmentation to the exactly true or non-falsity idea.

Algebraically $\mathbf{ETL}_1$ and $\mathbf{NFL}_1$ represent pre-ordered structures that can be equipped with the corresponding $\leq$-based semantics in the sense of Humberstone [18, p. 246]. It should be observed that here we can have at most meet-semilattice, which raises a question of suitable algebraic modeling for these logics. Again, the bi-consequence systems provide an opportunity for elaborating interesting algebraic structures with more than one ordering relation. Bi-lattices based on the four Belnapian truth values are generally well known and well studied, see, e.g. [14]. The algebraic structures that should correspond to $\mathbf{ETL}_1^2$ and $\mathbf{NFL}_1^2$ would be equipped with two ordering relations—a lattice ordering for $\vdash_1$ and a semi-lattice ordering for $\vdash_2$. The resulting lattices are worth studying.

Distinguishing T as the *designated* truth value and F as the *anti-designated* one makes it also possible to extend the range of types of entailment relations under

[17] Cf. an extension of the exactly true logic with an implicational connective in [39, pp. 458-459].

consideration, by applying the methodology of quasi-matrices initiated in [21]. The n-valued q-matrix (*quasi*-matrix) is defined there as a structure $\langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle$, where $\mathcal{V}$ is a non-empty set of values with at least two elements, $\mathcal{D}^+$ (the set of designated values) and $\mathcal{D}^-$ (the set of antidesignated values) are *disjoint* non-empty proper subsets of $\mathcal{V}$, and every f_c is a function on $\mathcal{V}$ with the same arity as c.

In [34] the Belnap generalized q-matrix B_4^* with $\{T, B\}$ and $\{F, B\}$ as the sets of designated and anti-designated truth values, respectively, was introduced. We may consider the *exact* Belnap generalized q-matrix B_4^e, which is the four-valued q-matrix based on Belnap's four truth values with $\mathcal{D}^+ = \{T\}$ and $\mathcal{D}^- = \{F\}$, and functions f_c defined as in the usual Belnap matrix. We can define then the following four entailment relations (cf. [34, p. 136]):

$$A \models_t B \text{ iff } \forall v : v(A) \in \mathcal{D}^+ \Rightarrow v(B) \in \mathcal{D}^+ \tag{7}$$

$$A \models_f B \text{ iff } \forall v : v(A) \notin \mathcal{D}^- \Rightarrow v(B) \notin \mathcal{D}^- \tag{8}$$

$$A \models_q B \text{ iff } \forall v : v(A) \notin \mathcal{D}^- \Rightarrow v(B) \in \mathcal{D}^+ \tag{9}$$

$$A \models_p B \text{ iff } \forall v : v(A) \in \mathcal{D}^+ \Rightarrow v(B) \notin \mathcal{D}^- \tag{10}$$

Conditions 7 and 8 are just another formulations of Definitions 7 and 8. Conditions 9 and 10 determine the relations of "quasi-consequence" (q-consequence) [21], and "plausibility-consequence" (p-consequence) [16] defined on the basis of B_4^e. We leave the study of these two latter relations for future work.

Acknowledgments We thank two anonymous reviewers for their valuable comments that greatly contributed to improving the final version of the paper.

References

1. Albuquerque, H., Přenosil, A., Rivieccio, U. (2017). An algebraic view of super-Belnap logics. *Studia Logica*, *105*, 1051–1086.
2. Ackermann, W. (1956). Begründung einer strengen Implikation. *Journal of Symbolic Logic*, *21*, 113–128.
3. Anderson, A.R., & Belnap, N.D. (1975). *Entailment: the logic of relevance and necessity* (Vol. I). Princeton: Princeton University Press.
4. Anderson, A.R., Belnap, N.D., Dunn, J.M. (1992). *Entailment: the logic of relevance and necessity* (Vol. II). Princeton: Princeton University Press.
5. Avron, A., & E. Orłowska (2003). Classical Gentzen-type methods in propositional many-valued logics. In M. Fitting (Ed.), *Beyond two: theory and applications of multiple-valued logic* (pp. 117–155). Berlin–Heidelberg: Springer.
6. Belnap, N.D. (1977). A useful four-valued logic. In J.M. Dunn, & G. Epstein (Eds.), *Modern uses of multiple-valued logic* (pp. 8–37). Dordrecht: D. Reidel Publishing Company.
7. Belnap, N.D. (1977). How a computer should think. In G. Ryle (Ed.), *Contemporary aspects of philosophy* (pp. 30–55). London: Oriel Press.
8. Demri, S., & Orlowska, E. (2002). *Incomplete information: structure, inference, complexity*. Berlin: Springer.
9. Dunn, J.M. (1976). Intuitive semantics for first-degree entailment and coupled trees. *Philosophical Studies*, *29*, 149–168.
10. Dunn, J.M. (1995). Positive modal logic. *Studia Logica*, *55*, 301–317.
11. Dunn, J.M. (1999). A comparative study of various model-theoretic treatmets of negation: a history of formal negation. In D.M. Gabbay, & H. Wansing (Eds.), *What is negation?* (pp. 23–51). Dordrecht/Boston/London: Kluwer Academic Publishers.
12. Dunn, J.M. (2000). Partiality and its dual. *Studia Logica*, *66*, 5–40.

13. Dunn, J.M., & Hardegree, G.M. (2001). *Algebraic methods in philosophical logic*. Oxford, New York: Clarendon Press, Oxford University Press.
14. Fitting, M.C. (1989). Bilattices and the theory of truth. *Journal of Philosophical Logic*, *18*, 225–256.
15. Font, J.M. (1997). Belnap's four-valued logic and De Morgan lattices. *Logic Journal of the IGPL*, *5*, 413–440.
16. Frankowski, S. (2004). Formalization of a plausible inference. *Bulletin of the Section of Logic*, *33*, 41–52.
17. Hartonas, C. (2017). Order-dual relational semantics for non-distributive propositional logics. *Logic Journal of the IGPL*, *25*, 145–182.
18. Humberstone, L. (2011). *The connectives*. Cambridge: MIT Press.
19. Labuschagne, W., Heidema, J., Britz, K. (2013). Supraclassical consequence relations. In S. Cranefield, & A. Nayak (Eds.), *AI 2013: advances in artificial intelligence. AI 2013. Lecture Notes in Computer Science* (Vol. 8272, pp. 326–337). Cham: Springer.
20. MacIntosh, J.J. (1991). Adverbially qualified truth values. *Pacific Philosophical Quarterly*, *72*, 131–142.
21. Malinowski, G. (1990). Q-consequence operation. *Reports on Mathematical Logic*, *24*, 49–59.
22. Marcos, J. (2011). The value of the two values. In J.-Y. Beziau, M.E. Coniglio (Eds.), *Logic without frontiers: festschrift for Walter Alexandre Carnielli on the occasion of his 60th birthday* (pp. 277–294). (Tributes), College Publications.
23. Odintsov, S., & Wansing, H. (2015). The logic of generalized truth-values and the logic of bilattices. *Studia Logica*, *103*, 91–112.
24. Omori, H., Wansing, H. (Eds.) (2017). *40 Years of FDE, Special Issue of Studia Logica*, *105*, Issue 6.
25. Pietz, A., & Rivieccio, U. (2013). Nothing but the truth. *Journal of Philosophical Logic*, *42*, 125–135.
26. Priest, G. (2008). *An introduction to non-classical logic* (2nd edn). Cambridge: Cambridge University Press.
27. Rescher, N. (1965). An intuitive interpretation of systems of four-valued logic. *Notre Dame Journal of Formal Logic*, *6*, 154–156.
28. Rivieccio, U. (2012). An infinity of super-Belnap logics. *Journal of Applied Non-Classical Logics*, *22*, 319–335.
29. Shramko, Y. (2016). Truth, falsehood, information and beyond: the American plan generalized. In K. Bimbo (Ed.), *J. Michael Dunn on Information Based Logics. Outstanding Contributions to Logic*, (Vol. 8, pp. 191–212). Berlin: Springer.
30. Shramko, Y. (2018). First-degree entailment and structural reasoning, to appear. In H. Omori, H. Wansing (Eds.), *New Essays on Belnap-Dunn Logic*, Synthese Library.
31. Shramko, Y. (2018). Dual-Belnal logic and anaything but falsehood, to appear in: *Journal of Applied Logic*.
32. Shramko, Y., Dunn, J.M., Takenaka, T. (2001). The trilaticce of constructive truth-values. *Journal of Logic and Computation*, *11*, 761–788.
33. Shramko, Y., & Wansing, H. (2005). Some useful sixteen-valued logics: how a computer network should think. *Journal of Philosophical Logic*, *34*, 121–153.
34. Shramko, Y., & Wansing, H. (2007). Entailment relations and/as truth values. *Bulletin of the Section of Logic*, *36*, 131–143.
35. Shramko, Y., Zaitsev, D., Belikov, A. (2017). First-degree entailment and its relatives. *Studia Logica*, *105*, 1291–1317.
36. Vakarelov, D. (1995). A duality between Pawlak's knowledge representation systems and bi-consequence systems. *Studia Logica*, *55*, 205–228.
37. van Benthem, J. (2008). Logic and reasoning: do the facts matter? *Studia Logica*, *88*, 67–84.
38. Wansing, H., & Shramko, Y. (2008). A note on two ways of defining a many-valued logic. In M. Pelis (Ed.) *The logica yearbook 2007* (pp. 255–266). Filosofia: Prague.
39. Wintein, S., & Muskens, R. (2016). A Gentzen calculus for nothing but the truth. *Journal of Philosophical Logic*, *45*, 451–465.
40. Zaitsev, D. (2009). A few more useful 8-valued logics for reasoning with tetralattice $EIGHT_4$. *Studia Logica*, *92*, 265–280.