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INVESTIGATION OF THE MOTION OF A HEAVY BODY OF REVOLUTION ON A PERFECTLY ROUGH PLANE BY THE KOVACIC ALGORITHM

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Abstract. Investigation of various problems of mechanics and mathematical physics is reduced to the solution of second-order linear differential equations with variable coefficients. In 1986, the American mathematician J. Kovacic proposed an algorithm for solution of a second-order linear differential equation in the case where the solution can be expressed in terms of so-called Liouville functions. If a linear second-order differential equation has no Liouville solutions, the Kovacic algorithm also allows one to ascertain this fact. In this paper, we discuss the application of the Kovacic algorithm to the problem of the motion of a heavy body of revolution on a perfectly rough horizontal plane. The existence of Liouville solutions of the problem is examined for the cases where the rolling body is an infinitely thin disk, a disk of finite thickness, a dynamically symmetric torus, a paraboloid of revolution, and a spindle-shaped body.

***Keywords and phrases*:** nonholonomic system, dynamically symmetric body, Kovacic algorithm, Liouville solutions.

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CONTENTS

1. Introduction

Problems of the motion of bodies that touch fixed or moving rigid surfaces have a long history. It is closely related to the process of formation and development of a large branch of analytical mechanic, namely, dynamics of nonholonomic systems. In works of I. Newton, L. Euler, I. Bernoulli, J. D'Alembert, and J. Lagrange, some problems on the rolling of rigid bodies without sliding were studied; these problems are typical in the dynamics of systems with nonholonomic constraints and hence they are considered as classical problems of nonholonomic mechanics. One of such classical problems is the problem on the motion of a heavy, rotationally symmetric rigid body on a fixed, perfectly rough horizontal plane. For the first time, this problem was considered in [27] by E. Lindelöf.

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To solve this problem, Lindelöf used the Hamilton principle or the Lagrange equations of the second kind obtained from this principle. Having written two equations of nonholonomic constraints, he applied them to the construction of the expression for the kinetic energy and erroneously assumed that the nonholonomicity of this problem is completely accounted and therefore the Lagrange equations of the second kind can be constructed. Naturally, the system of differential equations obtained by Londelöf was simpler than the valid system and admitted solutions in quadratures. The error was first detected by S. A. Chaplygin who informed Lindelöf about it. On October 25, 1895, Chaplygin reported his results on this topic in a session of the Physics Division of the Society of Devotees of Natural Science, Anthropology, and Ethnography. Chaplygin noted that "in the first pages of his work, ... E. Lindelöf made a serious mistake which led to the fact that the equations obtained turned out simpler than the actual valid equations, which explained the seeming achievement of the author." In this report, Chaplygin first presented his equations of the motion of nonholonomic systems. Two years later, he found a valid solution to the E. Lindelöf problem and published new results in [5]. In this paper, Chaplygin proved the integrability of this problem and detected that its solution is reduced to integration of a second-order linear differential equation whose coefficients depend on the shape and the mass distribution of the body. Having the general solution of the corresponding equation, the problem is reduced to quadratures. Chaplygin also found two cases where the general solution of the equation can be obtained. In the case where the body is a nonhomogeneous, dynamically symmetric ball, the general solution of the corresponding equation is expressed in terms of elementary functions (see [5]). In the case of the motion of a circular disk or a hoop on a horizontal plane, the general solution is expressed in terms of hypergeometric series (see [5]; this fact was also proved by P. Appell [1] and D. Korteweg [19]). In 1932, Kh. M. Mushtari continued to examine the problem on the motion of a heavy, rotationally symmetric body on a perfectly rough horizontal plane (see [35]). Under an additional condition imposed on the shape and the mass distribution of the body, two new particular cases were found in which the motion of the body can be described completely. In the first case, the moving body is bounded by the surface formed by rotating a parabolic arc about an axis passing through its focus, and in the second case, the moving rigid body is a rotationally symmetric paraboloid. Further development of Mustari's results was made by A. S. Kuleshov (see [22–25]). For any other rotationally symmetric bodies moving without sliding on a horizontal plane, the exact solution of the corresponding second-order linear differential equation is unknown. Therefore, it is interesting to find this solution for bodies different from the mentioned above (a ball or a disk) and hence to solve this problem completely. For this purpose, it is possible to apply the so-called Kovacic algorithm. In 1986, the American mathematician J. Kovacic presented in [21] an algorithm for finding a general solution of a second-order linear differential equation with variable coefficients for the case where this solution can be expressed in terms of so-called Liouville functions (see [15, 17, 21]). Recall that Liouville functions are functions constructed from rational functions by algebraic operations, taking exponentials, and integration. If a linear differential equation has no Liouville solutions, the Kovacic algorithm also allows one to ascertain this fact. The necessary condition for application of the Kovacic algorithm to a second-order linear differential equation is that the coefficients of this equation should be rational functions of independent variable. In this paper, the Kovacic algorithm is applied to the problem of motion of a heavy rotationally symmetric rigid body on a fixed perfectly rough horizontal plane. The paper is organized as follows. The Introduction contains a review of the results concerning the problem obtained in different years. In Sec. 2, we discuss the theoretical foundations of the Kovacic algorithm and describe the algorithm itself. We also discuss specific features of application of the Kovacic algorithm to second-order linear differential equations. In Sec. 3, we present a detailed formulation of the general problem of motion of a rotationally symmetric body on a perfectly rough horizontal plane. We derive the second-order linear differential equation for a rotationally symmetric body of an arbitrary shape. Further in Sec. 3 we consider two particular cases where the moving body is an infinitely thin circular disk or a circular disk of finite thickness. For both these cases, we derive the corresponding second-order linear differential equation and examine it using the Kovacic algorithm. As a result, we prove the nonexistence of Liouville solutions for both cases: in the problem of the motion of an infinitely thin circular disk and a circular disk of finite thickness rolling on a perfectly rough horizontal plane. In Sec. 4, we study the problem of the motion of a dynamically symmetric torus on a perfectly rough horizontal plane. Using the Kovacic algorithm, we prove the nonexistence of Liouville solutions in this problem for almost all values of parameters of the problem. Section 5 is devoted to the study of the motion of a dynamically symmetric paraboloid on a perfectly rough horizontal plane. Using the Kovacic algorithm, we prove that the general solution of the corresponding second-order linear differential equation is expressed through Liouville functions for all values of parameters of the problem. This fact allows us to study the qualitative behavior of the paraboloid on the plane. As a result, we obtain that the trajectory of the contact point on the surface of the paraboloid is a curve constructed from periodically repeated waves touching two parallels of paraboloid, whereas the trajectory of the contact point on the supporting plane is a similar curve lying between two concentric circles that are touched by the contact point alternately while the paraboloid moves on the plane. Similar results were obtained earlier by N. K. Moschuk (see [33, 34]). We also describe all steady motions of the paraboloid on the plane (permanent rotations and regular precessions) and examine their stability. In Sec. 6, we consider the problem of motion of a spindle-shaped body on a perfectly rough plane. This problem was earlier studied by Kh. M. Mushtari (see [35]). Direct application of the Kovacic algorithm to this problem allows one to state that the problem has no Liouville solutions for almost all values of parameters of the problem, except the case where these parameters satisfy the Mushtari condition (see [35]). Finally, we give short conclusions to summarize the content of the paper and discuss the future work. A part of results presented in this paper were discussed previously in [6–10, 26].

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2. Kovacic Algorithm and Its Theoretical Foundations

Investigation of many problems of mechanics and mathematical physics is reduced to the solution of second-order linear differential equations with variable coefficients. In 1986, the American mathematician J. Kovacic presented an algorithm for finding general solutions of a second-order linear homogeneous differential equations with rational coefficients in the case where these solutions can be expressed in terms of so-called Liouville functions (see [17, 21]). If a linear differential equation has no Liouville solutions, the Kovacic algorithm also allows one to ascertain that fact. Since the major part of the results of the paper was obtained by using the Kovacic algorithm, in this section we briefly discuss the algorithm itself and specific features of its application to second-order linear differential equations.

2.1. Statement of the problem. We consider the differential field $\mathbb{C}(x)$ of rational functions of a single (complex) variable x. We search for solutions of the differential equation

$$z'' + a(x)z' + b(x)z = 0, \tag{2.1}$$

where $a(x), b(x) \in \mathbb{C}(x)$. We are interested in so-called Liouville solution of Eq. (2.1). Recall that a solution is called a Liouville solution if it is an element of a Liouville field defined as follows.

Definition 1. Let F be a differential field of functions of a single complex variable x containing $\mathbb{C}(x)$, i.e., F is a field of characteristic zero with a differentiation operation $(\cdot)'$ possessing the following two properties:

$$(a+b)' = a' + b' \quad \text{and} \quad (ab)' = a'b + ab' \quad \text{for all } a \in F \text{ and } b \in F.$$

The field F is called a Liouville field if there exists a sequence (tower) of finite extensions of fields

$$\mathbb{C}(x) = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = F$$

obtained by joining a single element such that

$$F_i = F_{i-1}(\alpha), \quad \text{where} \quad \frac{\alpha'}{\alpha} \in F_{i-1} \quad \text{for each } i = 1, 2, \ldots, n$$

(i.e., F_i is generated by joining of the exponent of the indefinite integral over F_{i-1}), or

$$F_i = F_{i-1}(\alpha), \quad \text{where} \quad \alpha' \in F_{i-1}$$

(i.e., F_i is generated by joining the indefinite integral over F_{i-1}), or F_i is a finite algebraic extension over F_{i-1} (i.e., $F_i = F_{i-1}(\alpha)$ and α satisfies a finite-degree polynomial equation of the form

$$a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0,$$

where $a_j \in F_{i-1}$, $j = 0, 1, 2, \ldots, n$.

Thus, Liouville solutions are constructed from rational functions by algebraic operations, taking exponentials, and integration. In this way, we can get logarithmic or trigonometric functions but not special functions like the Bessel functions, the Legendre polynomials, or the Gauss hypergeometric functions. Applying the algorithm, it suffices to find only one Liouville solution of Eq. (2.1) since another solution cam be found as follows. Assume that $z_2 = \nu z_1$, where z_1 is the known solution and ν is some function to be determined. Using the differential equation (2.1) we obtain the following differential equation for ν:

$$z_1 \frac{d^2\nu}{dx^2} + \left(2\frac{dz_1}{dx} + a(x)z_1\right)\frac{d\nu}{dx} = 0,$$

which yields the second solution z_2:

$$z_2 = z_1 \int \left(\frac{1}{z_1^2} \exp\left(-\int a(x)dx\right)\right) dx.$$

If z_1 is a Liouville solution, then, clearly, the second solution z_2 is also a Liouville solution and hence all solutions of Eq. (2.1) are Liouville solution (since all solutions are linear combinations of z_1 and z_2). In order to reduce the original differential equation to a simpler form, we perform the substitution

$$y(x) = z(x) \exp\left(\frac{1}{2}\int a(x)dx\right). \tag{2.2}$$

Then Eq. (2.1) becomes

$$y'' + \left(b - \frac{1}{4}a^2 - \frac{1}{2}a'\right) y = 0$$

or

$$y'' = r(x)y, \quad r(x) = \frac{1}{2}a' + \frac{1}{4}a^2 - b. \tag{2.3}$$

Note that this change of variables does not change the Liouville nature of the solutions. If solutions of (2.1) are Liouville solutions, then solutions of (2.3) will also be Liouville solutions. Further, we will assume that the equation considered has the form (2.3) and $r(x) \in \mathbb{C}(x)$, $r(x) \notin \mathbb{C}$. Using the inverse transformation

$$z(x) = y(x) \exp\left(-\frac{1}{2}\int a(x)dx\right)$$

we can transform solutions of (2.3) to solutions of (2.1).

2.2. Preliminaries. In this section, we discuss some facts from the theory of linear differential equations that form the basis of the Kovacic algorithm. A part of them is presented with detailed proofs, another part is presented without proofs, but proofs can be found in the references. We start from the description of the possible structure of the solution of the differential equation (2.3).

2.2.1. The four cases. The following theorem by J. Kovacic [21] determines the structure of solutions with which the algorithm deals.

Theorem 1. *For the differential equation* (2.3), *the following four cases can occur.*

1. *The differential equation* (2.3) *has a solution of the form* $\eta = \exp\int \omega(x)dx$, *where* $\omega(x) \in \mathbb{C}(x)$.
2. *Equation* (2.3) *has a solution of the form* $\eta = \exp\int \omega(x)dx$, *where* $\omega(x)$ *is algebraic function of degree* 2 *over* $\mathbb{C}(x)$ *and Case* 1 *does not hold.*
3. *All solutions of Eq.* (2.3) *are algebraic functions over* $\mathbb{C}(x)$ *and Cases* 1 *and* 2 *do not hold. In this case, the solutions of Eq.* (2.3) *have the form* $\eta = \exp\int \omega(x)dx$, *where* $\omega(x)$ *is an algebraic function of degree* 4, 6, *or* 12 *over* $\mathbb{C}(x)$.
4. *Equation* (2.3) *has no Liouville solutions.*

Below we discuss the basic ideas of the proof of this theorem. Let η and ζ be two independent solutions of the differential equation (2.3). Denote by $\overline{G}$ the differential extension field of $\mathbb{C}(x)$ generated by η and ζ, i.e., $\overline{G} = \mathbb{C}(x)\,(\eta, \eta', \zeta, \zeta')$. Higher derivatives of η and ζ are not needed since $\eta'' = r\eta \in \overline{G}$, $\eta''' = r'\eta + r\eta' \in \overline{G}$, etc. The *Galois group* $G = G\left(\overline{G}/\mathbb{C}(x)\right)$ of the differential equation (2.3) is the Galois group of $\overline{G}$ over $\mathbb{C}(x)$. In other words, G is the group of all differential automorphisms of $\overline{G}$ leaving elements of $\mathbb{C}(x)$ fixed. Recall that an automorphisms of a group H is an isomorphism from H to itself and a differential automorphism is an automorphism that commutes with the differentiation operation $(\cdot)'$. This means that G is the group of all automorphisms $\sigma : \overline{G} \to \overline{G}$ such that $\sigma(a') = (\sigma a)'$ for all $a \in \overline{G}$ and $\sigma f = f$ for all $f \in \mathbb{C}(x)$. The Galois group G of the differential equation (2.3) is isomorphic to a subgroup of the group $\mathrm{GL}(2, \mathbb{C})$ of all invertible (2×2)-matrices with complex coefficients, i.e., each $\sigma \in G$ corresponds to a matrix

$$\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix},$$

where a_σ, b_σ, c_σ, and d_σ are elements of $\mathbb{C}$. This correspondence is established as follows. Since η and ζ are solutions of Eq. (2.3) and any $\sigma \in G$ is an differential automorphism, we have

$$(\sigma\eta)'' = \sigma(\eta'') = \sigma(r\eta) = \sigma r \cdot \sigma\eta = r\sigma\eta$$

and hence $\sigma\eta$ is also a solution of Eq. (2.3). Further, $\sigma\eta$ is a linear combination of η and ζ since any solution of Eq. (2.3) is a linear combination of any two independent solutions of (2.3). Then we can write

$$\sigma\eta = a_\sigma\eta + b_\sigma\zeta, \quad a_\sigma, b_\sigma \in \mathbb{C}.$$

Similarly we obtain

$$\sigma\zeta = c_\sigma\eta + d_\sigma\zeta, \quad c_\sigma, d_\sigma \in \mathbb{C}.$$

Combining these two results we have

$$\begin{pmatrix} \sigma\eta \\ \sigma\zeta \end{pmatrix} = \begin{pmatrix} a_\sigma\eta + b_\sigma\zeta \\ c_\sigma\eta + d_\sigma\zeta \end{pmatrix} = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \begin{pmatrix} \eta \\ \zeta \end{pmatrix};$$

obviously, σ corresponds to the matrix

$$\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}.$$

Using the Wronskian of the solutions η and ζ, we can show that the Galois group G of Eq. (2.3) is isomorphic to the subgroup $\mathrm{SL}(2, \mathbb{C})$ of the group $\mathrm{GL}(2, \mathbb{C})$ consisting of invertible (2×2)-matrices with determinant 1. (Recall that the Wronskian W of η and ζ is by definition $W = \eta\zeta' - \eta'\zeta$.) The derivative of W vanishes:

$$W' = \eta'\zeta' + \eta\zeta'' - \eta'\zeta' - \eta''\zeta = \eta\zeta'' - \eta''\zeta = \eta r\zeta - r\eta\zeta = 0.$$

Hence the Wronskian W of η and ζ is a constant and hence for any $\sigma \in G$ we have $\sigma W = W$ (since $W \in \mathbb{C}(x)$ and σ, by definition, leaves $\mathbb{C}(x)$ fixed). This implies that

$$\begin{aligned}\sigma W = \sigma\big(\eta\zeta' - \eta'\zeta\big) &= \sigma\eta(\sigma\zeta)' - (\sigma\eta)'\sigma\zeta \\ &= \big(a_\sigma\eta + b_\sigma\zeta\big)\big(c_\sigma\eta' + d_\sigma\zeta'\big) - \big(a_\sigma\eta' + b_\sigma\zeta'\big)\big(c_\sigma\eta + d_\sigma\zeta\big) \\ &= \big(a_\sigma d_\sigma - b_\sigma c_\sigma\big)\big(\eta\zeta' - \eta'\zeta\big) = \big(a_\sigma d_\sigma - b_\sigma c_\sigma\big)W\end{aligned}$$

and hence

$$a_\sigma d_\sigma - b_\sigma c_\sigma = 1.$$

The following two facts are presented without proofs.

Theorem 2. *The Galois group G of Eq.* (2.3) *is isomorphic to an algebraic subgroup of* $\mathrm{SL}(2,\mathbb{C})$.

This theorem is a fundamental fact from the Picard–Vessiot theory. Its proof can be found in [21]. Recall that a subgroup K of the group $\mathrm{GL}(2,\mathbb{C})$ is said to be an *algebraic group* if there exist a finite number of polynomials $P_1, \dots, P_n$, where $P_i \in \mathbb{C}[x_1, x_2, x_3, x_4]$ (the polynomial ring with 4 variables x_1, x_2, x_3, x_4 over the field $\mathbb{C}$), such that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of K if and only if

$$P_1(a,b,c,d) = \cdots = P_n(a,b,c,d) = 0.$$

Further, for any algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$, the following lemma holds.

Lemma 1 (see [15, 21]). *If G is an algebraic subgroup of* $\mathrm{SL}(2,\mathbb{C})$, *then one of the following four cases can occur*:

1. *G is triangulable, i.e., there exists $x \in G$ such that for any $g \in G$, the matrix xgx^{-1} is a triangular matrix. We assume that xgx^{-1} is a lower triangular matrix*:

$$\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix},$$

where $a, b \in \mathbb{C}$. Recall that G is a subgroup of $\mathrm{SL}(2,\mathbb{C})$ *and hence the determinant of xgx^{-1} is equal to* 1.

2. *G is conjugate to a subgroup of the group $D^\dagger$, where*

$$D^\dagger = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix},\ c \in \mathbb{C},\ c \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix},\ c \in \mathbb{C},\ c \neq 0 \right\}$$

and Case 1 *does not hold, i.e., there exists $x \in G$ such that for any $g \in G$, xgx^{-1} is either a diagonal matrix or an anti-diagonal matrix, but there is no $x \in G$ such that for all $g \in G$ the matrix xgx^{-1} is lower triangular (this case includes only strictly diagonal matrices).*

3. *G is finite algebraic subgroup and Cases* 1 *and* 2 *do not hold.*

4. *$G = \mathrm{SL}(2,\mathbb{C})$, i.e., G is the infinite group of all (2×2)-matrices with determinant* 1.

Thus, we know that the Galois group G of Eq. (2.3) is isomorphic to an algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$. We also know that any algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$ satisfies the above lemma. Now we can apply the lemma to the Galois group of Eq. (2.3) and establish the relationship betwenn various subgroups of the group $\mathrm{SL}(2,\mathbb{C})$ and solutions of Eq. (2.3) listed in Theorem 1. In Case 1, the group G is triangulable. Assume that an element $x \in G$ has been found and each matrix is conjugated to a lower triangular matrix (this is equivalent to a change of the basis in the vector space or to the choice of two different independent solutions $\bar{\eta}$ and $\bar{\zeta}$). Then each element $\sigma \in G$ has the form

$$\begin{pmatrix} a_\sigma & 0 \\ c_\sigma & a_\sigma^{-1} \end{pmatrix}, \quad a_\sigma, c_\sigma \in \mathbb{C},$$

and maps η to $\sigma\eta = a_\sigma \eta$. Setting $\omega = \eta'/\eta$ or, equivalently, $\eta = \exp\int \omega(x)dx$, we have

$$\sigma\omega = \sigma\left(\frac{\eta'}{\eta}\right) = \frac{(\sigma\eta)'}{\sigma\eta} = \frac{a_\sigma\eta'}{a_\sigma\eta} = \frac{\eta'}{\eta} = \omega$$

and hence $\omega \in \mathbb{C}(x)$. This is Case 1 of Theorem 1: Eq. (2.3) has a solution of the form $\eta = \exp\int \omega(x)dx$, where $\omega(x) \in \mathbb{C}(x)$. In Case 2, the group G is conjugate to a subgroup of the group $D^\dagger$. In this case, any element of G has one of the following forms:

$$\begin{pmatrix} a_\sigma & 0 \\ 0 & a_\sigma^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & b_\sigma \\ -b_\sigma^{-1} & 0 \end{pmatrix},$$

so either ($\sigma\eta = a_\sigma\eta$ and $\sigma\zeta = a_\sigma^{-1}\zeta$) or ($\sigma\eta = b_\sigma\zeta$ and $\sigma\zeta = -b_\sigma^{-1}\eta$). Note that in both cases we have $\sigma\left(\eta^2\zeta^2\right) = \eta^2\zeta^2$, so that $\eta^2\zeta^2 \in \mathbb{C}(x)$. If we set now $\omega = \eta'/\eta$ (i.e., $\eta = \exp\int\omega(x)dx$) and $\varphi = \zeta'/\zeta$, then either ($\sigma\omega = \omega$ and $\sigma\varphi = \varphi$) or ($\sigma\omega = \varphi$ and $\sigma\varphi = \omega$). Minimally, both cases are described by the conditions $\sigma^2\omega = \omega$ or $\sigma^2\omega - \omega = 0$, so ω satisfies a polynomial equation of degree 2 over $\mathbb{C}(x)$, hence it is algebraic function of degree 2 over $\mathbb{C}(x)$. This is Case 2 of Theorem 1. In Case 3, the group G is a finite group, i.e., there are only a finite number of automorphisms $\sigma_1, \dots, \sigma_n$. Consider an arbitrary elementary symmetric function of the arguments $\sigma_1\eta$, $\sigma_2\eta$, …, $\sigma_n\eta$, for example,

$$\sum \sigma_i\eta = \sigma_1\eta + \sigma_2\eta + \cdots + \sigma_n\eta.$$

For any $\sigma_j \in G$ we have

$$\sigma_j\left(\sum \sigma_i\eta\right) = \sum \sigma_i\eta$$

since $\sigma_i\sigma_j \in G$ for all σ_i (because G is a group and hence is closed). Hence $\sum \sigma_i\eta = f(x) \in \mathbb{C}(x)$ and the solution η satisfies the equation

$$\sigma_1\eta + \sigma_2\eta + \cdots + \sigma_n\eta - f(x) = 0;$$

i.e., η is an algebraic function over $\mathbb{C}(x)$. Similar arguments hold for ζ. Therefore, η and ζ are algebraic over $\mathbb{C}(x)$ and hence all solutions of Eq. (2.3) are algebraic over $\mathbb{C}(x)$. To clarify the structure of the group G in Case 3, we present the following theorem (its detailed proof can be found in [21]).

Theorem 3. *If K is a finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$, *then one of the following possibilities is realized*:
(1) *K is conjugate to a subgroup of the group $D^\dagger$*;
(2) *K has order* 24;
(3) *K has order* 48;
(4) *K has order* 120.

Clearly, the first case of this theorem is a particular subcase of Case 2 of Lemma 1. This means that in Case 3, the group G has order 24, 48, or 120, and hence the order of η over $\mathbb{C}(x)$ is 24, 48, or 120, respectively. In each of these cases, the functions of solutions η and ζ belonging to $\mathbb{C}(x)$ are known: if the group G has order 24, then

$$(\eta^4 + 8\eta\zeta^3)^3 \in \mathbb{C}(x),$$

if the group G has order 48, then

$$(\eta^5\zeta - \eta\zeta^5)^2 \in \mathbb{C}(x),$$

and if the group G has order 120, then

$$\eta^{11}\zeta - 11\eta^6\zeta^6 - \eta\zeta^{11} \in \mathbb{C}(x)$$

(for details, see [21]). In Case 4 of Lemma 1, we have $G = \mathrm{SL}(2, \mathbb{C})$. Below we show that in this case, Eq. (2.3) has no Liouville solution. We prove this assertion by the contrary. Assume that Eq. (2.3) has a Liouville solution. Then the second solution obtained by the method of order reduction (see above) is also a Liouville solution, and hence all solutions of Eq. (2.3) are Liouville solutions (since any solution of Eq. (2.3) is a linear combination of two independent solutions). Clearly, $\overline{G} = \mathbb{C}(x)(\eta, \eta', \zeta, \zeta')$ is contained in a Liouville extension and the component G^0 of the identity of the group G must be solvable (see [18, p. 415]). (Recall that the component of the identity of a group is the largest connected subgroup of the group containing the identity. A set is said to be connected if any two points in the set can be joined by an arc lying in the set.)

A group H is said to be *solvable* (in the sense of the Galois theory) if

$$H = H_0 \supset H_1 \supset \ldots \supset H_m = \{e\},$$

where each H_{i+1} is normal in H_i, each factor group H_i/H_{i+1} is abelian, and e is the identity element of H. If $G = \mathrm{SL}(2, \mathbb{C})$, then $G^0 = \mathrm{SL}(2, \mathbb{C})$ and hence $\mathrm{SL}\,(2, \mathbb{C})$ must be solvable. But $\mathrm{SL}(2, \mathbb{C})$ is not solvable; this contradiction implies that the initial assumption was false and Eq. (2.3) has no Liouville solutions. This is Case 4 of Theorem 1.

2.2.2. Necessary conditions. In order to reduce the amount of calculations involved in the solution of Eq. (2.3), Kovacic (see [21]) indicated some conditions on the function r in the right-hand side of the equation. For each of the first three cases where Liouville solutions exist, these conditions are different. For example, if the function r satisfies the conditions corresponding to Case 1 of Theorem 1, then we must search for solutions of Eq. (2.3) exactly in the form indicated for this case. If the function r does not satisfy any conditions corresponding the Cases 1–2 of Theorem 1, then we conclude that Eq. (2.3) has no Liouville solutions. These conditions are necessary but not sufficient. For example, if the conditions corresponding to Case 1 of Theorem 1 are violated, then we must turn to the verification of the conditions corresponding to Cases 2 and 3. If these conditions are fulfilled, then we must search for solutions of Eq. (2.3) exactly in the form indicated for the corresponding case. However, the existence of such a solution is not guaranteed. In order to explain the sense of the necessary conditions mentioned, we recall some facts from complex analysis. Recall that any analytic function f of a complex variable z can be expanded in a Laurent series in a neighborhood of any point a as follows:

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \ldots.$$

The part of this series

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$$

containing nonnegative powers of $z - a$ is called the *analytic part* of the Laurent series whereas the other part, namely,

$$\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \ldots,$$

is called the *principal part* of the expansion. By definition, a point a is called a *pole* of $f(z)$ of order n if the proncipal part of the Laurent expansion contains a finite number of terms and the last term has the form $a_{-n}/(z-a)^n$. If $f(z)$ is a rational function of z, then a point a is a pole of $f(z)$ of order n if it is a root of the denominator of $f(z)$ of multiplicity n. Let $z=\infty$ be a zero of a function $f(z)$ of order n (i.e., n is the order of the pole at $z=0$ of $f(z)$). Then we say that n is the order of $f(z)$ at infinity. If $f(z)$ is a rational function, then its order at $z=\infty$ is the difference between the degrees of the denominator and the numerator. The following theorem states necessary conditions under which the first three cases listed in Theorem 1 can hold.

Theorem 4. *For the differential equation* (2.3), *the following conditions are necessary for the existence of Liouville solutions in the corresponding case from Theorem* 1.

1. *Each pole of the function r has order* 1 *or an even order. The order of r at ∞ is even or greater than* 2.
2. *The function r has at least one pole whose order either* 2 *or an odd number greater than* 2.
3. *The function r has no poles of order greater than* 2. *The order of r at ∞ is at least* 2. *If the partial-fraction expansion of r is*

$$r=\sum_i \frac{\alpha_i}{(x-c_i)^2}+\sum_j \frac{\beta_j}{x-d_j},$$

then

$$\sqrt{1+4\alpha_i}\in\mathbb{Q}\quad \textit{for each } i,\qquad \sum_j \beta_j=0$$

and, moreover,

$$\sqrt{1+4\gamma}\in\mathbb{Q},\qquad \textit{where}\quad \gamma=\sum_i \alpha_i+\sum_j \beta_j d_j.$$

Below we present a sketch of the proof of this theorem. Some ideas will be explained in more details in the description of the algorithm itself (see Sec. 2.3). In Case 1, the differential equation (2.3) has a solution of the form

$$\eta=\exp\int\omega(x)dx,\quad \omega(x)\in\mathbb{C}(x). \tag{2.4}$$

Substituting this solution into Eq. (2.3), we see that the function $\omega(x)$ satisfies the differential equation

$$\omega'+\omega^2=r. \tag{2.5}$$

Since both functions $r(x)$ and $\omega(x)$ belong to $\mathbb{C}(x)$, they can be expanded in Laurent series in a neighborhood of a point c of the complex plane as follows:

$$\omega=b(x-c)^\mu+\text{higher-order terms},\quad \mu\in\mathbb{Z},\ b\neq 0, \tag{2.6}$$

$$r=\alpha(x-c)^\nu+\text{higher-order terms},\quad \nu\in\mathbb{Z},\ \alpha\neq 0. \tag{2.7}$$

Substituting (2.6) and (2.7) to (2.5) we obtain

$$\mu b(x-c)^{\mu-1}+\cdots+b^2(x-c)^{2\mu}+\cdots=\alpha(x-c)^\nu+\dots. \tag{2.8}$$

We prove demonstrate that if c is a pole of r (i.e., $\nu<0$), then its order is either 1 or even. In the expansion (2.8), we indicate only the lowest powers of $x-c$ in each term. These terms must cancel. Indeed,

(1) if $\nu=-1$ (i.e., c is a pole of order 1), then $\mu=-1$ and we can cancel the two terms in the left-hand side of the expansion (2.8);
(2) if $\nu=-2$, then $\mu=-1$, and we can cancel three terms of the lowest degree;

(3) if $\nu \leq -3$, then the corresponding coefficients of the lowest power of $x - c$ in the expansion (2.8) yield

$$\nu \geq \min(\mu - 1, 2\mu).$$

If $\nu \leq -3$, this implies that $\mu < -1$, i.e., $2\mu < \mu - 1$. Since $b \neq 0$ (by assumption), we have $\nu = 2\mu$, i.e., ν is an even number as required.

These arguments also show that if r has a pole of order $-\nu = -2\mu \geq 4$ at c, then ω has a pole of order $-\mu$ at c. This fact will be used in the proof of the algorithm in Sec. 2.3.2. The verification of the conditions on the order of r at $x = \infty$ is similar; it is based on the expansions of r and ω at $x = \infty$ (see [21] for details). In Case 2, the differential equation (2.3) has a solution of the form

$$\eta = \exp \int \omega(x)dx, \tag{2.9}$$

where $\omega(x)$ is an algebraic function of degree 2 over $\mathbb{C}(x)$. The Galois group G of the differential equation (2.3) is conjugate to a subgroup of the group $D^\dagger$, so that for every $\sigma \in G$, either ($\sigma\eta = a_\sigma\eta$ and $\sigma\zeta = a_\sigma^{-1}\zeta$) or ($\sigma\eta = b_\sigma\zeta$ and $\sigma\zeta = -b_\sigma^{-1}\eta$). In both cases, we have $\sigma(\eta^2\zeta^2) = \eta^2\zeta^2$, so that $\eta^2\zeta^2 \in \mathbb{C}(x)$. Also, $\eta\zeta \notin \mathbb{C}(x)$ since in the opposite case we have $\sigma(\eta\zeta) = \eta\zeta = a_\sigma\eta a_\sigma^{-1}\zeta$, and G would be consists of diagonal matrices with a_σ and a_σ^{-1} on the diagonal (i.e., the case where $\sigma\eta = b_\sigma\zeta$ and $\sigma\zeta = -b_\sigma^{-1}\eta$ will be impossible). Therefore, we can represent $\eta^2\zeta^2$ in the form $\prod(x - c_i)^{e_i}$, $e_i \in \mathbb{Z}$, where at least one of e_i is be odd (if all e_i are odd, then we have $\eta\zeta \in \mathbb{C}(x)$, which is impossible). We assume that $\eta^2\zeta^2 = (x - c)^e \prod(x - c_i)^{e_i}$, where e is an odd number. Let

$$\varphi = \frac{(\eta\zeta)'}{\eta\zeta} = \frac{\frac{1}{2}(\eta^2\zeta^2)'}{\eta^2\zeta^2}.$$

Since $\eta'' = r\eta$ and $\zeta'' = r\zeta$, it is easy to find by a direct calculation that the function φ satisfies the differential equation

$$\varphi'' + 3\varphi\varphi' + \varphi^3 = 4r\varphi + 2r'. \tag{2.10}$$

Expand the functions r and φ in Laurent series in a neighborhood of c:

$$\varphi = \frac{e}{2(x - c)} + \text{polynomial in } x - c, \tag{2.11}$$

$$r = \alpha(x - c)^\nu + \text{higher-order terms}. \tag{2.12}$$

Substituting (2.11) and (2.12) into the differential equation (2.10), we obtain

$$\frac{e}{(x-c)^3} + \dots + \frac{-\frac{3}{4}e^2}{(x-c)^3} + \dots + \frac{\frac{1}{8}e^3}{(x-c)^3} + \dots = 2\alpha(e + \nu)(x - c)^{\nu - 1} + \dots.$$

If $\nu > -2$, then

$$e - \frac{3}{4}e^2 + \frac{1}{8}e^3 = 0$$

and hence $e = 0, 2, 4$. However, e must be odd, so $\nu \leq -2$. If $\nu < -2$, then $2\alpha(e + \nu) = 0$ and $e = -\nu$, so that ν is odd. Therefore, either $\nu = -2$ or $\nu < -2$ is odd, i.e., $r(x)$ has either a pole of order 2 or a pole of odd order greater than 2. In Case 3, the differential equation (2.3) has a solution of the form

$$\eta = \exp \int \omega(x)dx, \tag{2.13}$$

where $\omega(x)$ is an algebraic function of degree 4, 6, or 12 over $\mathbb{C}(x)$, i.e., η is an algebraic function over $\mathbb{C}(x)$. It can be expanded in a Puiseux series (a series with fractional exponents) in a neighborhood of a certain point c in the complex plane. Since η is a solution of the differential equation (2.3), we have

$$\eta'' = r\eta. \tag{2.14}$$

Expanding the functions η and r in a neighborhood of c, we obtain

$$\eta = a(x-c)^{\mu} + \text{higher-order terms}, \quad a \in \mathbb{C},\ a \neq 0,\ \mu \in \mathbb{Q}, \tag{2.15}$$

$$r = \alpha(x-c)^{\nu} + \text{higher-order terms}, \quad \alpha \in \mathbb{C},\ \alpha \neq 0,\ \nu \in \mathbb{Z}. \tag{2.16}$$

Substituting (2.15) and (2.16) into (2.14), we obtain

$$a\mu(\mu-1)(x-c)^{\mu-2} + \cdots = \alpha a(x-c)^{\mu+\nu} + \dots. \tag{2.17}$$

The lowest-order term in the right-hand side is the product of the lowest-order terms of η and r. It cannot be zero, so we have $\mu+\nu \geq \mu-2$, i.e., $\nu \geq -2$; therefore, the orders of poles of the functions r can be only 1 or 2. If $\nu = -2$, then, equating the coefficients of $(x-c)^{\mu-2}$ in both sides of (2.17), we obtain

$$\alpha = \mu(\mu-1) \quad \text{or} \quad \mu = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\alpha}.$$

Since $\mu \in \mathbb{Q}$ by assumption, we have $\sqrt{1+4\alpha} \in \mathbb{Q}$ and the partial-fraction expansion of r is

$$r = \sum_i \frac{\alpha_i}{(x-c_i)^2} + \sum_j \frac{\beta_j}{x-d_j} + \text{polynomial}$$

and $\sqrt{1+4\alpha_i} \in \mathbb{Q}$ for each i. The remaining conditions of Case 3 can be obtained similarly, namely, by expanding r and η in a neighborhood of $x = \infty$ and substituting the corresponding series in Eq. (2.14).

2.3. Kovacic algorithm and its proof.

2.3.1. Kovacic algorithm for Case 1. The goal of the Kovacic algorithm is to find a solution of the differential equation (2.3) in the form $\eta = P \exp \int \theta(x)dx$, where $P \in \mathbb{C}[x]$ is a polynomial with coefficients in the field of complex numbers $\mathbb{C}$ and $\theta \in \mathbb{C}(x)$. Since η can be written as

$$\eta = \exp \int \left(\frac{P'}{P} + \theta \right) dx,$$

this corresponds to the general form of a solution in Case 1 described by Theorem 1, where $\omega = P'/P + \theta$. The first step of the algorithm consists of determining parts of the partial fraction expansion of θ. In the second step, we add these parts and obtain a function, which is a candidate for the role of θ. The maximal number of possible candidates is $2^{\rho+1}$, where ρ is the number of poles of r. If there are no candidates, then Case 1 cannot hold. In the third step, for each candidate for θ, we try to find a suitable polynomial P. If this is possible, then we obtain a desired solution of the differential equation (2.3); otherwise, Case 1 cannot hold. We assume that the necessary conditions (see Sec. 2.2.2) for Case 1 are fulfilled and denote by Γ the set of finite poles of the function r.

Step 1. For each $c \in \Gamma \cup \{\infty\}$, we introduce the rational function $[\sqrt{r}]_c$ and two complex numbers α_c^+ and α_c^- as described below.

(c_1) If $c \in \Gamma$ and c is a pole of order 1, then

$$[\sqrt{r}]_c = 0, \quad \alpha_c^+ = \alpha_c^- = 1.$$

(c_2) If $c \in \Gamma$ and c is a pole of order 2, then

$$[\sqrt{r}]_c = 0.$$

Let b be the coefficient of $(x-c)^{-2}$ in the partial fraction expansion of r. Then

$$\alpha_c^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}.$$

(c_3) If $c \in \Gamma$ and c is a pole of order $2\nu \geq 4$ (the order must be even due to the necessary conditions stated in Sec. 2.2.2), then $[\sqrt{r}]_c$ is the sum of terms involving $(x-c)^{-i}$, $2 \leq i \leq \nu$, in the Laurent expansion of $\sqrt{r}$ at c. There are two possibilities for $[\sqrt{r}]_c$ that differ by sign; we can choose one of them. Thus,

$$[\sqrt{r}]_c = \frac{a}{(x-c)^\nu} + \cdots + \frac{d}{(x-c)^2}. \tag{2.18}$$

In practice, one would not construct the Laurent series for $\sqrt{r}$ in a neighborhood of c: it suffices to find the function $[\sqrt{r}]_c$ by the method of undefined coefficients. Let b be the coefficient of $(x-c)^{-\nu-1}$ in $r - [\sqrt{r}]_c^2$. Then

$$\alpha_c^\pm = \frac{1}{2}\left(\pm\frac{b}{a} + \nu\right).$$

(∞_1) If the order of the function r at $x = \infty$ is greater than 2, then

$$[\sqrt{r}]_\infty = 0, \quad \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1.$$

(∞_2) If the order of r at $x = \infty$ is 2, then

$$[\sqrt{r}]_\infty = 0.$$

Let b be the coefficient of x^{-2} in the Laurent expansion of r in a neighborhood of $x = \infty$. If $r = s/t$, where $s \in \mathbb{C}[x]$ and $t \in \mathbb{C}[x]$ are relatively prime polynomials, then b is the ratio of the leading coefficients of s and t. Then

$$\alpha_\infty^\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}.$$

(∞_3) If the order of r at $x = \infty$ is $-2\nu \leq 0$ (it is even due to the necessary conditions stated in Sec. 2.2.2), then the function $[\sqrt{r}]_\infty$ is the sum of terms involving x^i, $0 \leq i \leq \nu$, of the Laurent expansion of $\sqrt{r}$ at $x = \infty$ (one of the two possibilities can be chosen). Thus,

$$[\sqrt{r}]_\infty = ax^\nu + \cdots + d.$$

Let b be the coefficient of $x^{\nu-1}$ in $r - \left([\sqrt{r}]_\infty\right)^2$. Then

$$\alpha_\infty^\pm = \frac{1}{2}\left(\pm\frac{b}{a} - \nu\right).$$

Step 2. For each tuple $s = (s(c))_{c \in \Gamma \cup \{\infty\}}$, where $s(c)$ is $+1$ or -1, let

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}. \tag{2.19}$$

If d is a nonnegative integer, then the function

$$\theta = \sum_{c \in \Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty \tag{2.20}$$

is a candidate for $\theta(x)$. If d is not a nonnegative integer, then the corresponding tuple s should be rejected. If all tuples s have been rejected, then Case 1 cannot hold.

Step 3. This step must be executed for each of the tuples s found on Step 2. If for a certain tuple success is achieved, then a solution is found; otherwise, this tuple is rejected. If all tuples found on Step 2 have been rejected, then Case 1 cannot hold. For each tuple s found on Step 2, we search for a polynomial P of degree d (the constant d is defined by the formula (2.19)) satisfying the differential equation

$$P'' + 2\theta P' + (\theta' + \theta^2 - r)P = 0; \tag{2.21}$$

the method of undefined coefficients is convenient for this purpose. If such a polynomial exists, then

$$\eta = P \exp \int \theta(x) dx$$

is a solution of the differential equation (2.3). If for each tuple s found on Step 2, we cannot find such a polynomial, then Case 1 cannot hold.

Below, we present the proof of the Kovacic algorithm for Case 1 of the differential equation (2.3).

2.3.2. Proof of the Kovacic algorithm for Case 1. In Case 1, we search for a solution of the differential equation (2.3) in the form (2.4), where

$$\omega(x) = \theta(x) + \frac{P'(x)}{P(x)}, \quad \theta(x) \in \mathbb{C}(x), \quad P(x) \in \mathbb{C}[x].$$

Since $\omega(x) \in \mathbb{C}(x)$, it can be expanded in a Laurent series in a neighborhood of any point of the complex plane. The algorithm starts from the determining the partial fraction expansion of $\omega(x)$ by using the Laurent expansion of r and the Riccati equation (2.5). We can write the Laurent expansion of ω in a neighborhood of a pole c of the function r as follows:

$$\omega = \sum_{i=2}^{\mu} \frac{a_i}{(x-c)^i} + \frac{e_c}{x-c} + \sum_{j=0}^{\infty} b_j (x-c)^j.$$

In the sequel, we will not need to determine the explicit form of a_i and b_j, so we indroduce the notation

$$[\omega]_c = \sum_{i=2}^{\mu} \frac{a_i}{(x-c)^i}, \quad \bar{\omega}_c = \sum_{j=0}^{\infty} b_j (x-c)^j.$$

Then

$$\omega = \sum_{i=2}^{\mu} \frac{a_i}{(x-c)^i} + \frac{e_c}{x-c} + \sum_{j=0}^{\infty} b_j (x-c)^j = [\omega]_c + \frac{e_c}{x-c} + \bar{\omega}_c. \tag{2.22}$$

Now the main task of the algorithm is to determine these parts of the function ω, i.e., find e_c and $[\omega]_c$ and the polynomial remainder $\bar{\omega}_c$. We know that, due to the necessary conditions for Case 1, all poles of the function r have order either 1, or 2, or an even order greater than 4. First, we assume that c is a pole of the function r of order 1. Then

$$r = \frac{\alpha}{x-c} + \text{polynomial in } x - c. \tag{2.23}$$

Substituting (2.22) and (2.23) into the Riccati equation (2.5), we obtain

$$-\frac{\mu a_\mu}{(x-c)^{\mu+1}} + \dots + \frac{a_\mu^2}{(x-c)^{2\mu}} + \dots = \frac{\alpha}{x-c} + \dots.$$

If we assume that $a_\mu \neq 0$ and $\mu \geq 2$, then $2\mu \geq 4$ and the term $\dfrac{a_\mu^2}{(x-c)^{2\mu}}$ cannot be canceled with any other term of the latter equation. Therefore, $[\omega]_c = 0$ and

$$\omega = \frac{e_c}{x-c} + \bar{\omega}_c.$$

Using this expression and substituting it into the Riccati equation (2.5) again, we get

$$-\frac{e_c}{(x-c)^2} + \bar{\omega}_c' + \frac{e_c^2}{(x-c)^2} + \frac{2e_c\bar{\omega}_c}{x-c} + \bar{\omega}_c^2 = \frac{\alpha}{x-c} + \dots.$$

The term with $(x-c)^{-2}$ must vanish, so we have $-e_c + e_c^2 = 0$, i.e., $e_c = 0$ or 1. The case $e_c = 0$ is impossible since in this case the right-hand side of the Riccati equation has no poles whereas the right-hand side has a pole of order 1. Hence, if c is a pole of r of order 1, then ω has the form

$$\omega = \frac{e_c}{x-c} + \bar{\omega}_c, \quad e_c = 1.$$

Now we assume that c is a pole of r of order 2. Then

$$r = \frac{b}{(x-c)^2} + \frac{\alpha}{x-c} + \dots. \tag{2.24}$$

Substituting (2.22) and (2.24) into the Riccati equation (2.5), we obtain

$$-\frac{\mu a_\mu}{(x-c)^{\mu+1}} + \dots + \frac{a_\mu^2}{(x-c)^{2\mu}} + \dots = \frac{b}{(x-c)^2} + \frac{\alpha}{x-c} + \dots.$$

As above, if we assume that $a_\mu \neq 0$ and $\mu \geq 2$, i.e., $2\mu \geq 4$, then the term $a_\mu^2/(x-c)^{2\mu}$ cannot be canceled with any other term of the latter equation. Therefore, $[\omega]_c = 0$ and

$$\omega = \frac{e_c}{x-c} + \bar{\omega}_c.$$

Substituting this expression into the Riccati equation (2.5) we get

$$-\frac{e_c}{(x-c)^2} + \bar{\omega}_c' + \frac{e_c^2}{(x-c)^2} + \frac{2e_c\bar{\omega}_c}{x-c} + \bar{\omega}_c^2 = \frac{b}{(x-c)^2} + \frac{\alpha}{x-c} + \dots.$$

Equating the coefficient of $1/(x-c)^2$ to zero and taking into account this relation, we obtain $e_c^2 - e_c = b$, i.e., for e_c we have the following two possibilities:

$$e_c = \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \qquad \text{or} \quad e_c = \frac{1}{2} - \frac{1}{2}\sqrt{1+4b}.$$

Hence, if c is a pole of r of order 2, then

$$\omega = \frac{e_c}{x-c} + \bar{\omega}_c, \quad e_c = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}.$$

Now we assume that c is a pole of the function r of order $2\nu \geq 4$. From the proof of the necessary conditions for Case 1 (see Sec. 2.2.2) we see that ω must have a pole of order ν at c, i.e.,

$$[\omega]_c = \sum_{i=2}^{\nu} \frac{a_i}{(x-c)^i}.$$

According to the formulation of the algorithm (see Sec. 2.3.1), we define the function $[\sqrt{r}]_c$ by the formula (2.18). Introduce the notation

$$\bar{r}_c = \sqrt{r} - [\sqrt{r}]_c;$$

then

$$r = \left(\bar{r}_c + [\sqrt{r}]_c\right)^2 = \bar{r}_c^2 + 2\bar{r}_c[\sqrt{r}]_c + \left([\sqrt{r}]_c\right)^2$$

and, therefore,

$$r - \left([\sqrt{r}]_c\right)^2 = \bar{r}_c^2 + 2\bar{r}_c[\sqrt{r}]_c. \tag{2.25}$$

Using (2.22) and the Riccati equation (2.5), we can show that

$$\begin{aligned}\left([\omega]_c + [\sqrt{r}]_c\right)\left([\omega]_c - [\sqrt{r}]_c\right) &= \left([\omega]_c\right)^2 - \left([\sqrt{r}]_c\right)^2 \\ &= -[\omega]_c' + \frac{e_c}{(x-c)^2} - \bar{\omega}_c' + r - \left([\sqrt{r}]_c\right)^2 - \frac{2e_c[\omega]_c}{x-c} - \frac{e_c^2}{(x-c)^2} - \frac{2e_c\bar{\omega}_c}{x-c} - 2[\omega]_c\bar{\omega}_c - \bar{\omega}_c^2.\end{aligned}$$

The left-hand side of this equation contains only terms of the form $1/(x-c)^i$, where $i = \nu+2, \dots, 2\nu$. The right-hand side contains terms involving $1/(x-c)^i$, $i = 1, \dots, \nu+1$, and polynomials in $x-c$. Since there are no terms involving $1/(x-c)^i$ in the right-hand side for $i = \nu+2, \dots, 2\nu$, the left-hand side must be equal to zero (we take into account the inequality $\nu \geq 2$) and hence either $[\omega]_c = [\sqrt{r}]_c$ or $[\omega]_c = -[\sqrt{r}]_c$. Finally,

$$\omega = \pm[\sqrt{r}]_c + \frac{e_c}{x-c} + \bar{\omega}_c.$$

Using this representation of the function ω and substituting it in the Riccati equation (2.5), we obtain

$$\frac{\pm a\nu}{(x-c)^{\nu+1}} + \dots + \frac{e_c}{(x-c)^2} - \bar{\omega}_c' + \frac{b}{(x-c)^{\nu+1}} + \dots$$
$$+ \frac{\mp 2ae_c}{(x-c)^{\nu+1}} - \frac{e_c^2}{(x-c)^2} - \frac{2e_c\bar{\omega}_c}{x-c} \mp \frac{2\bar{\omega}_c a}{(x-c)^{\nu}} + \dots = 0.$$

Equating the coefficients of $(x-c)^{-\nu-1}$ on both sides, we obtain

$$\pm a\nu + b \mp 2ae_c = 0,$$

and hence

$$e_c = \frac{1}{2}\left(\nu + \frac{b}{a}\right) \quad \text{or} \quad e_c = \frac{1}{2}\left(\nu - \frac{b}{a}\right).$$

Therefore, if c is a pole of the function r of even order $2\nu \geq 4$, then

$$\omega = \pm[\sqrt{r}]_c + \frac{e_c}{x-c} + \bar{\omega}_c, \quad e_c = \frac{1}{2}\left(\nu \pm \frac{b}{a}\right).$$

Now we consider a point g of the complex plain, which is not a pole of r. The Laurent expansion of r at this point is a polynomial in $x-g$. Expanding ω in a neighborhood of g and using the Riccati equation (2.5), we obtain

$$\omega = \frac{f}{x-g} + \text{polynomial in } x-g,$$

where $f = 0$ or 1. Finally, we have

$$\omega = [\omega]_c + \frac{e_c}{x-c} + \bar{\omega}_c = \sum_{c\in\Gamma}\left(\frac{e_c}{x-c} \pm [\sqrt{r}]_c\right) + \sum_{i=1}^{d}\frac{1}{x-g_i} + R,$$

where $[\sqrt{r}]_c = 0$ if c is not a pole of r of order ≥ 4 and R is a polynomial in $\mathbb{C}[x]$. We now determine the polynomial part R of the function ω. We use the Laurent expansion of ω in a neighborhood of $x = \infty$:

$$\omega = R + \frac{e_\infty}{x} + \text{lower powers of } x. \tag{2.26}$$

Using arguments similar to the above, we arrive at the following results. If the order of the pole $x = \infty$ of the function r is greater than 2, then $e_\infty = 0$ or 1 and $R = 0$; if r has a pole of order 2 at $x = \infty$, then

$$e_\infty = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}, \quad R = 0,$$

and if the order of the pole of r at $x = \infty$ is equal to $-2\nu \leq 0$, then

$$e_\infty = \frac{1}{2}\left(-\nu \pm \frac{b}{a}\right), \quad R = \pm[\sqrt{r}]_\infty.$$

Therefore,

$$\omega = \sum_{c\in\Gamma}\left(\frac{e_c}{x-c} + s(c)[\sqrt{r}]_c\right) + s(\infty)[\sqrt{r}]_\infty + \sum_{i=1}^{d}\frac{1}{x-g_i}, \tag{2.27}$$

where $s(c)$ is $+1$ or -1 depending on the sign of the corresponding e_c and $s(\infty)$ is $+1$ or -1 depending on the sign of e_∞. Moreover, $[\sqrt{r}]_c = 0$ if c is not a pole of r of order ≥ 4 and $[\sqrt{r}]_\infty = 0$ if the order of the pole at r is ≥ 2. Expanding (2.26) in a neighborhood of the point $x = \infty$ and equating it to (2.27), we obtain the equation

$$e_\infty = \sum_{c \in \Gamma} e_c + \sum_{i=1}^{d} 1$$

and hence we arrive at the following expression for d in terms of e_∞ and e_c:

$$d = e_\infty - \sum_{c \in \Gamma} e_c.$$

If we set $P = \prod_{i=1}^{d} (x - g_i)$ (note that d is the degree of P), then

$$\frac{P'}{P} = \sum_{i=1}^{d} \frac{1}{x - g_i},$$

and if

$$\theta = \sum_{c \in \Gamma} \left(\frac{e_c}{x - c} \pm [\sqrt{r}]_c \right) \pm [\sqrt{r}]_\infty,$$

then $\omega = \theta + P'/P$. Thus, we have complete information on θ is known; it remains to find an explicit form of P. Substitute the expression $\omega = \theta + P'/P$ into the Riccati equation (2.5), we obtain

$$\omega' = \theta' + \frac{PP'' - P'^2}{P^2}, \quad \omega^2 = \theta^2 + \frac{2\theta P'}{P} + \frac{P'^2}{P^2},$$
$$P'' + 2\theta P' + (\theta' + \theta^2 - r)P = 0.$$

We see that if ω satisfies the Riccati equation (2.5), then P satisfies the differential equation (2.21). We can verify that if P satisfies (2.21), then ω satisfies the Riccati equation (2.5) and hence the function $\eta = \exp \int \omega(x) dx$ satisfies the differential equation (2.3). Indeed,

$$\omega' + \omega^2 = \theta' + \frac{PP'' - P'^2}{P^2} + \theta^2 + \frac{2\theta P'}{P} + \frac{P'^2}{P^2} = \frac{P'' + 2\theta P' + P(\theta' + \theta^2)}{P} = \frac{Pr}{P} = r.$$

This completes the proof of the validity of the Kovacic algorithm for Case 1.

2.3.3. The Kovacic algorithm for Case 2. Considering the Kovacic algorithm for Case 2, we assume that the necessary conditions for this case (see Sec. 2.2.2) are fulfilled, and that Case 1 is known to fail. As in Case 1, we first collect data for each finite pole c of the function r and also for the pole of R at infinity. For each of the poles, we form the set E_c (or E_∞) consisting of integers (their number may vary from 1 to 3). Next we consider tuples of elements of these sets; after analysis, some of these tuples will be rejected. If all tuples have been rejected, then Case 2 cannot hold for the differential equation (2.3). For each suitable tuple, we search for a polynomial that satisfies a certain linear differential equation. If such a polynomial exists for a certain tuple, then a solution of the differential equation (2.3) has been found. If for all tuples, there are no such polynomials, then Case 2 cannot hold for the differential equation (2.3). Now we describe the algorithm for Case 2. Let Γ be the set of finite poles of the function r.

Step 1. For each $c \in \Gamma \cup \{\infty\}$, we define E_c as follows.

(c_1) If $c \in \Gamma$ is a pole of the function r of order 1, then

$$E_c = \{4\}.$$

(c_2) If $c \in \Gamma$ is a pole of the function r of order 2 and b is the coefficient of $1/(x-c)^2$ in the partial fraction expansion of r, then

$$E_c = \big\{(2 + k\sqrt{1+4b}) \cap \mathbb{Z},\ k = 0, \pm 2\big\}.$$

(c_3) If $c \in \Gamma$ is a pole of the function r of order $\nu > 2$, then

$$E_c = \{\nu\}.$$

(∞_1) If the function r has order >2 at the point $x = \infty$, then

$$E_\infty = \{0, 2, 4\}.$$

(∞_2) If the function r has order 2 at the point $x = \infty$ and b is the coefficient of $1/x^2$ in the Laurent series expansion of r at $x = \infty$, then

$$E_\infty = \big\{(2 + k\sqrt{1+4b}) \cap \mathbb{Z},\ k = 0, \pm 2\big\}.$$

(∞_3) If the order of r at $x = \infty$ is $\nu < 2$, then

$$E_\infty = \{\nu\}.$$

Step 2. We consider all tuples $s = (e_\infty, e_c)$, $c \in \Gamma$, where $e_c \in E_c$, $e_\infty \in E_\infty$ and at least one of these numbers is odd. Let

$$d = \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right). \tag{2.28}$$

If d is a nonnegative integer, then the corresponding tuple is suitable; otherwise it must be rejected.

Step 3. For each suitable tuple obtained on Step 2, we construct the rational function

$$\theta = \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c}. \tag{2.29}$$

Next we search for a polynomial P of degree d (where d is defined by the formula (2.28)) such that

$$P''' + 3\theta P'' + \big(3\theta^2 + 3\theta' - 4r\big)P' + \big(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\big)P = 0. \tag{2.30}$$

If success is achieved and such a polynomial is found, we set $\varphi = \theta + P'/P$, and let ω be a solution of the square equation (algebraic equation of degree 2)

$$\omega^2 - \varphi\omega + \frac{1}{2}\varphi' + \frac{1}{2}\varphi^2 - r = 0. \tag{2.31}$$

Then $\eta = \exp\int \omega(x)dx$ is a solution of the differential equation (2.3). If we cannot find such a polynomial for any suitable tuple found on Step 2, then Case 2 cannot hold for the differential equation (2.3).

2.3.4. Proof of the Kovacic algorithm for Case 2. In Case 2, we search for a solution to the differential equation (2.3) of the form (2.9). The Galois group of Eq. (2.3) is conjugate to a subgroup of the group

$$D^\dagger = \left\{\begin{pmatrix} c & 0 \\ 0 & c^{-1}\end{pmatrix},\ c \in \mathbb{C},\ c \neq 0\right\} \cup \left\{\begin{pmatrix} 0 & c \\ -c^{-1} & 0\end{pmatrix},\ c \in \mathbb{C},\ c \neq 0\right\}$$

and $\eta^2\zeta^2$ is an invariant of the group (i.e., $\eta^2\zeta^2$ is a fixed element under the action of automorphisms from the Galois group of Eq. (2.3)). Hence, $\eta^2\zeta^2 \in \mathbb{C}(x)$ but $\eta\zeta \notin \mathbb{C}(x)$ (otherwise we would have Case 1). Therefore, we can write

$$\eta^2\zeta^2 = \alpha \prod_{c\in\Gamma}(x-c)^{e_c} \prod_{i=1}^{m}(x-g_i)^{f_i}, \quad \alpha = \text{const},$$

and

$$\varphi = \frac{(\eta\zeta)'}{\eta\zeta} = \frac{1}{2}\frac{(\eta^2\zeta^2)'}{\eta^2\zeta^2} = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} + \frac{1}{2}\sum_{i=1}^{m}\frac{f_i}{x-g_i}. \tag{2.32}$$

The task of the algorithm is to find explicit values of e_c and f_i (we need not find g_i). If the function φ is determined, we can construct a quadratic equation with coefficients depending on φ that determines ω and hence a solution of Eq. (2.3). Since η and ζ are solutions of Eq. (2.3), we conclude that $\eta'' = r\eta$ and $\zeta'' = r\zeta$. Then φ satisfies Eq. (2.10). This equation allows one to find a relationship between the function φ (i.e., in fact, e_c and f_i) and the known function r. Now we can determine the coefficients e_c by analyzing the poles of the function r and the Laurent expansion of r and φ in neighborhoods of these poles. Assume that c is a pole of the function r of order 1; then

$$r = \frac{\alpha}{x-c} + \text{polynomial in } x-c \tag{2.33}$$

and

$$\varphi = \frac{1}{2}\frac{e_c}{x-c} + k + \text{polynomial in } x-c, \quad k \in \mathbb{C}. \tag{2.34}$$

Substituting (2.33) and (2.34) into Eq. (2.10), we obtain

$$\frac{e_c}{(x-c)^3} + \dots + \frac{-\frac{3}{4}e_c^2}{(x-c)^3} + \frac{-\frac{3}{2}e_c k}{(x-c)^2} + \dots + \frac{\frac{1}{8}e_c^3}{(x-c)^3} + \frac{\frac{3}{4}e_c^2 k}{(x-c)^2} + \dots$$
$$= \frac{2\alpha e_c}{(x-c)^2} + \dots + \frac{-2\alpha}{(x-c)^2} + \dots.$$

Equating the coefficient of $1/(x-c)^3$ to zero, we obtain

$$e_c - \frac{3}{4}e_c^2 + \frac{1}{8}e_c^3 = 0, \quad \text{i.e.,} \quad e_c = 0, 2, 4.$$

Equating the coefficients of $1/(x-c)^2$ on both sides, we obtain

$$-\frac{3}{2}e_c k + \frac{3}{4}e_c^2 k = 2\alpha e_c - 2\alpha.$$

Since $\alpha \neq 0$, we have $e_c \neq 0$ and $e_c \neq 2$. Hence if c is a pole of the function r of order 1, then $e_c = 4$. Now we assume that c is a pole of the function r of order 2. Then

$$r = \frac{b}{(x-c)^2} + \frac{\alpha}{x-c} + \text{polynomial in } x-c \tag{2.35}$$

and also

$$\varphi = \frac{1}{2}\frac{e_c}{x-c} + \text{polynomial in } x-c. \tag{2.36}$$

Substituting (2.35) and (2.36) into (2.10) we obtain

$$\frac{e_c}{(x-c)^3} + \dots + \frac{-\frac{3}{4}e_c^2}{(x-c)^3} + \dots + \frac{\frac{1}{8}e_c^3}{(x-c)^3} = \frac{2be_c}{(x-c)^3} + \dots + \frac{-4b}{(x-c)^3} + \dots.$$

Equating the coefficients of $1/(x-c)^3$ on both sides, we obtain

$$e_c - \frac{3}{4}e_c^2 + \frac{1}{8}e_c^3 = 2be_c - 4b$$

so that there exists three possibilities for e_c:

$$e_c = 2, \quad e_c = 2 + 2\sqrt{1+4b}, \quad e_c = 2 - 2\sqrt{1+4b}.$$

Since e_c must be an integer, we reject noninteger solutions for e_c. Finally, if c is a pole of r of order 2, then

$$e_c = 2, \quad 2 \pm 2\sqrt{1+4b} \in \mathbb{Z}.$$

Now assume that that c is a pole of r of order $\nu > 2$. Then

$$r = \frac{\alpha}{(x-c)^\nu} + \text{higher-order terms}, \tag{2.37}$$

and for the function φ we have the expression (2.36). Substitute (2.36) and (2.37) into Eq. (2.10), we obtain

$$\frac{e_c}{(x-c)^3} + \frac{-\frac{3}{4}e_c^2}{(x-c)^3} + \cdots + \frac{\frac{1}{8}e_c^3}{(x-c)^3} + \cdots = \frac{2\alpha e_c}{(x-c)^{\nu+1}} + \cdots + \frac{-2\alpha\nu}{(x-c)^{\nu+1}} + \cdots.$$

Since $\nu > 2$ and, therefore, $\nu + 1 > 3$, we have $2\alpha e_c - 2\alpha\nu = 0$, i.e., $e_c = \nu$. Hence if c is a pole of r of order $\nu > 2$, then $e_c = \nu$. Now consider the points g_i that are poles of φ but regular points of r. In this case

$$r \text{ is a polynomial in } x - g_i, \tag{2.38}$$

$$\varphi = \frac{1}{2}\frac{f_i}{x-g_i} + g + \text{polynomial in } x - g_i, \quad g \in \mathbb{C}. \tag{2.39}$$

Substituting (2.38) and (2.39) into (2.10), we get

$$\frac{f_i}{(x-g_i)^3} + \cdots + \frac{-\frac{3}{4}f_i^2}{(x-g_i)^3} + \frac{-\frac{3}{2}f_i g}{(x-g_i)^2} + \cdots + \frac{\frac{1}{8}f_i^3}{(x-g_i)^3} + \frac{\frac{3}{4}f_i^2 g}{(x-g_i)^2} + \cdots = \frac{\alpha}{x-g_i} + \cdots.$$

Since the right-hand side does not contain terms with $1/(x-g_i)^3$, we have

$$f_i - \frac{3}{4}f_i^2 + \frac{1}{8}f_i^3 = 0,$$

i.e., $f_i = 0$, 2, or 4; hence all f_i in φ are even. Finally, we obtain

$$\eta^2\zeta^2 = \alpha \prod_{c\in\Gamma}(x-c)^{e_c}P^2,$$

where $\alpha = \text{const}$ and

$$P^2 = \prod_{i=1}^{m}(x-g_i)^{f_i} \in \mathbb{C}[x].$$

Now we can use the expansion of φ in a neighborhood of $x = \infty$, namely,

$$\varphi = \frac{e_\infty}{2x} + \text{lower-order terms}. \tag{2.40}$$

Using arguments similar to the above, we obtain that $e_\infty = 0, 2, 4$ if the order of r at $x = \infty$ is greater than 2; $e_\infty = 2$ or $2 \pm 2\sqrt{1+4b}$ if the order of r at $x = \infty$ is 2, and $e_\infty = \nu$ if the order of r at $x = \infty$ is $\nu < 2$. Expanding the expression (2.32) into the Laurent series in a neighborhood of $x = \infty$, equating the result to (2.40), and comparing the coefficients of $1/x$ on both sides, we obtain

$$\frac{1}{2}e_\infty = \frac{1}{2}\sum_{c\in\Gamma} e_c + \frac{1}{2}\sum_{i=1}^{m} f_i.$$

If d is the degree of P, then $2d = \sum_{i=1}^{m} f_i$ so that from the last equation we obtain the following expression for d in terms of e_c and e_∞, which is similar (2.28):

$$d = \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right).$$

If we set

$$\theta = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c},$$

then $\varphi = \theta + P'/P$. Using this expression and Eq. (2.10), we obtain for P the differential equation (2.30). Nevertheless, we still do not have an expression for ω, the objective of the algorithm. Kovacic [21] introduced the algebraic equation (2.31) for ω. We can verify that if ω is a solution of Eq. (2.31) and Case 2 holds, then ω satisfies Eq. (2.5) and hence the function $\eta = \exp \int \omega(x)dx$ satisfies Eq. (2.3). Differentiating Eq. (2.31), we obtain

$$(2\omega - \varphi)\omega' = \varphi'\omega - \frac{1}{2}\varphi'' - \varphi\varphi' + r'.$$

On the other hand, from (2.31) we have

$$\omega^2 - r = \varphi\omega - \frac{1}{2}\varphi' - \frac{1}{2}\varphi^2,$$

so that

$$(2\omega - \varphi)(\omega' + \omega^2 - r) = -\frac{1}{2}\Big(\varphi'' + 3\varphi\varphi' + \varphi^3 - 4r\varphi - 2r'\Big) = 0$$

(see (2.10)); hence either $2\omega - \varphi = 0$ or $\omega' + \omega^2 - r = 0$. Now $2\omega - \varphi$ cannot be zero since in this case $\omega = \frac{1}{2}\varphi \in \mathbb{C}(x)$, which corresponds to Case 1. Hence $\omega' + \omega^2 - r = 0$ and $\eta = \exp \int \omega(x)dx$ is a solution of Eq. (2.3). This proves the validity of the algorithm for Case 2.

2.3.5. Kovacic algorithm for Case 3. In Case 3, the differential equation (2.3) has only algebraic solutions. We assume that Cases 1 and 2 are known to fail (although, in this case Eq. (2.3) may have algebraic solutions). Let η be a solution of Eq. (2.3) and $\omega = \eta'/\eta$. Then it is possible to prove (see [21]) that ω is an algebraic function over $\mathbb{C}(x)$ of degree 4, 6, or 12. In this case, the task of the algorithm is to find the minimal polynomial annihilating the function ω. The algorithm is described as follows. As above, we denote by Γ the set of finite poles of the function r. Recall that, due to the necessary conditions (see Sec. 2.2.2), the function r cannot have poles of order > 2.

Step 1. For each $c \in \Gamma \cup \{\infty\}$, we define the set E_c of integers as follows.

(c_1) If $c \in \Gamma$ is a pole of order 1, then $E_c = \{12\}$.

(c_2) If $c \in \Gamma$ is a pole of order 2 and b is the coefficient of $1/(x-c)^2$ in the partial fraction expansion of r, then

$$E_c = \big\{(6 + k\sqrt{1+4b}) \cap \mathbb{Z}\big\}, \quad k = 0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5,\ \pm6.$$

Note that by the necessary conditions, an analog of the case (c_3) in Step 1 cannot occur.

(∞) If the Laurent series for r at $x = \infty$ has the form $r = b/x^2 + \dots$, where $b \in \mathbb{C}$ and possibly $b = 0$, then

$$E_\infty = \big\{(6 + k\sqrt{1+4b}) \cap \mathbb{Z}\big\}, \quad k = 0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5,\ \pm6.$$

Step 2. Consider the tuples $s = (e_\infty, e_c)$, $c \in \Gamma$, where $e_c \in E_c$, $e_\infty \in E_\infty$. Let

$$d = e_\infty - \sum_{c\in\Gamma} e_c. \tag{2.41}$$

If d is a nonnegative integer, the tuple is suitable; otherwise, it must be rejected.

Step 3. For each suitable tuple obtained on Step 2, we construct the rational function

$$\theta = \sum_{c\in\Gamma} \frac{e_c}{x-c} \tag{2.42}$$

and the polynomial

$$S = \prod_{c\in\Gamma}(x-c). \tag{2.43}$$

We search for a polynomial P of degree d satisfying a certain differential equation, which can be written in the recursive form:

$$P_{12} = -P, \quad P_{i-1} = -SP_i' + ((12-i)S' - S\theta)P_i - (12-i)(i+1)S^2 r P_{i+1}, \quad P_{-1} = 0. \tag{2.44}$$

The second formula in (2.44) should be applied for $i = 12, \dots, 0$. The sense of the last equation is as follows: after P_{-1} has been calculated, we must equate it to zero. If success is achieved, then for any solution ω of the algebraic equation

$$\sum_{i=0}^{12} \frac{S^i P_i}{(12-i)!} \omega^i = 0,$$

the function $\eta = \exp \int \omega(x)dx$ is a solution of Eq. (2.3). Otherwise, Case 3 cannot hold for Eq. (2.3).

The proof of the correctness of the algorithm for Case 3 is similar to the proof for Case 2 presented above and we omit it for brevity (for details, see [21]. Thus, if the problem of the study of a mechanical system is reduced to the solution of a second-order linear differential equation, we can try to find a change of variables that reduces the coefficients of the initial equation to rational functions and then apply the Kovacic algorithm to find Liouville solutions of the equation obtained. Below, we apply the Kovacic algorithm to the study of the classical problem of nonholonomic system dynamics on the motion of a heavy, rigid, rotationally symmetric body on a fixed, perfectly rough horizontal plane.

3. General Problem on the Motion of a Rotationally Symmetric Body on a Perfectly Rough Plane. Motion of a Thin Circular Disk and a Circular Disk of Finite Thickness

3.1. Formulation of the problem.

3.1.1. Basic coordinate systems. Let a rigid body move on a fixed horizontal plane in a homogeneous gravity field. We assume that the body is bounded by a strictly convex surface, i.e., at every instant of time, there exists a unique contact point. In the majority of problems, we assume that the surface of the body has a unique tangent plane at the contact point. We also consider the motion of a body with a sharp edge in the case where the contact point lies of the edge. We introduce a fixed coordinate system $Oxyz$ whose plane Oxy coincides with the horizontal supporting plane and the Oz-axis is directed vertically upward. Let $\boldsymbol{\gamma}$ be the unit normal vector to the surface of the moving body at the point M of contact of the body with the horizontal plane (see Fig. 1).

Now we assume that the moving body is rotationally symmetric, i.e., it is bounded by a strictly convex surface of revolution and the axis of rotation of this surface coincides with the axis of dynamical symmetry of the body and contains the center of mass G of the body. We also introduce the coordinate system $Gx_1x_2x_3$ whose origin coincides with the center of mass G of the body and the axes coincide with the principal central axes of inertia of the body. Due to the rotational symmetry of the body, its axis of symmetry is one of the principal central axes of inertia of the body; let the Gx_3-axis be directed along the symmetry axis of the body (Fig. 2). The orientation of the body relative to the fixed coordinate system $Oxyz$ is defined by the Euler angles ψ, θ, and φ, where θ is the angle between the axis Gx_3 of dynamical symmetry of the body and the Oz-axis. Therefore, the mutual orientation of the coordinate systems $Oxyz$ and $Gx_1x_2x_3$ is defined by the matrix $\boldsymbol{A}$ of direction cosines:

	x_1	x_2	x_3
x	a_{11}	a_{12}	a_{13}
y	a_{21}	a_{22}	a_{23}
z	a_{31}	a_{32}	a_{33}

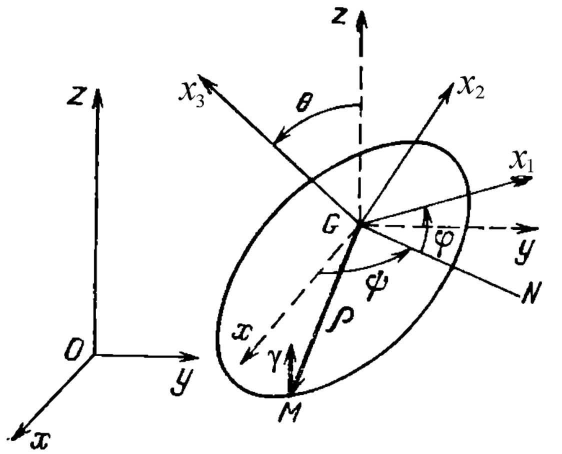


Fig. 1. Motion of a rotationally symmetric body: the Euler angles.

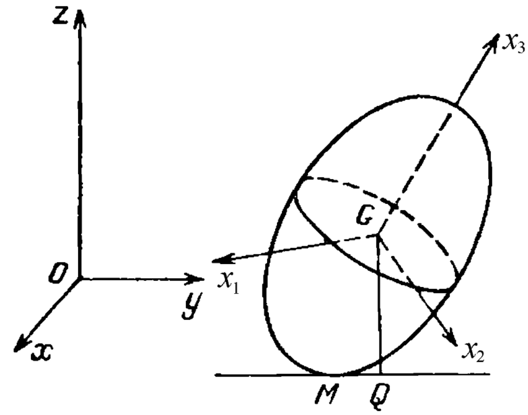


Fig. 2. Motion of a rotationally symmetric body: the principal central axes of inertia of the body.

The elements a_{ij} of this matrix are expressed through the Euler angles ψ, θ, and φ by the following formulas:

$$\begin{aligned}
&a_{11} = \cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta, && a_{12} = -\cos\psi\sin\varphi - \sin\psi\cos\varphi\cos\theta,\\
&a_{13} = \sin\psi\sin\theta, && a_{21} = \sin\psi\cos\varphi + \cos\psi\sin\varphi\cos\theta,\\
&a_{22} = -\sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta, && a_{23} = -\cos\psi\sin\theta,\\
&a_{31} = \sin\varphi\sin\theta, \quad a_{32} = \cos\varphi\sin\theta, \quad a_{33} = \cos\theta.
\end{aligned} \tag{3.1}$$

Let

$$F(x_1, x_2, x_3) = 0, \tag{3.2}$$

be the equation of the surface of the body in the coordinate system $Gx_1x_2x_3$. We choose the sign of $F(x_1, x_2, x_3)$ so that

$$\gamma = -\frac{\operatorname{grad} F}{|\operatorname{grad} F|},$$

where

$$\operatorname{grad} F = \frac{\partial F}{\partial x_1}\boldsymbol{e}_1 + \frac{\partial F}{\partial x_2}\boldsymbol{e}_2 + \frac{\partial F}{\partial x_3}\boldsymbol{e}_3, \quad |\operatorname{grad} F| = \sqrt{\left(\frac{\partial F}{\partial x_1}\right)^2 + \left(\frac{\partial F}{\partial x_2}\right)^2 + \left(\frac{\partial F}{\partial x_3}\right)^2},$$

and $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are the unit vectors of the axes Gx_1, Gx_2, and Gx_3, respectively. Using these equations and (3.1), we obtain

$$\begin{aligned}
a_{31} &= \sin\theta\sin\varphi = -\frac{1}{|\operatorname{grad} F|}\frac{\partial F}{\partial x_1},\\
a_{32} &= \sin\theta\cos\varphi = -\frac{1}{|\operatorname{grad} F|}\frac{\partial F}{\partial x_2},\\
a_{33} &= \cos\theta = -\frac{1}{|\operatorname{grad} F|}\frac{\partial F}{\partial x_3}.
\end{aligned} \tag{3.3}$$

Since the moving rigid body is rotationally (geometrically) symmetric, we can rewrite Eq. (3.2) in the form

$$F(\delta, x_3) = 0, \quad \delta = \sqrt{x_1^2 + x_2^2}. \tag{3.4}$$

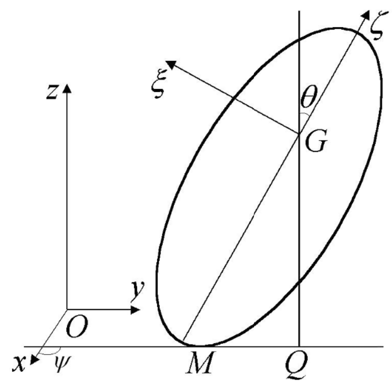


Fig. 3. Motion of a rotationally symmetric body: basic coordinate systems.

Equations (3.3) take the form

$$\begin{aligned} a_{31} &= \sin\theta\sin\varphi = -\frac{1}{|\operatorname{grad} F|}\frac{\partial F}{\partial \delta}\cdot\frac{x_1}{\delta}, \\ a_{32} &= \sin\theta\cos\varphi = -\frac{1}{|\operatorname{grad} F|}\frac{\partial F}{\partial \delta}\cdot\frac{x_2}{\delta}, \\ a_{33} &= \cos\theta = -\frac{1}{|\operatorname{grad} F|}\frac{\partial F}{\partial x_3}. \end{aligned} \tag{3.5}$$

The first two equations in (3.5) yields the identity

$$x_1\cos\varphi = x_2\sin\varphi. \tag{3.6}$$

Due to the dynamical symmetry of the body, the directions of the orthogonal axes Gx_1 and Gx_2 can be chosen up to rotation by an arbitrary angle in the equatorial plane of the central ellipsoid of inertia of the body. Let us choose these axes so that the first coordinate x_1 of the contact point M is $x_1 = -\delta$ when $\varphi = \pi/2$. Figure 3 shows the meridional section of the body corresponding to the angle $\varphi = \pi/2$.

From (3.4) and (3.6) we obtain that

$$x_1 = -\delta\sin\varphi, \quad x_2 = -\delta\cos\varphi. \tag{3.7}$$

Using (3.7) and the second and third equations (3.5) we get

$$\frac{\partial F}{\partial \delta}\cos\theta + \frac{\partial F}{\partial x_3}\sin\theta = 0.$$

This equation together with (3.4) shows that δ and x_3 are functions of θ. Now we introduce the coordinate system $G\xi\eta\zeta$ with the origin at the center of mass G of the body, which moves both in the space and in the body, such that the $G\zeta$-axis coincides with the Gx_3-axis, the $G\eta$-axis is directed along the vector product $[\boldsymbol{e}_3\times\boldsymbol{\gamma}]$, and the $G\xi$-axis is such that $G\xi\eta\zeta$ is a right orthogonal system (see Fig. 3). We denote the unit basis vectors of this coordinate system by $\boldsymbol{e}_\xi$, $\boldsymbol{e}_\eta$, and $\boldsymbol{e}_\zeta$, respectively. It is easy to understand that the $G\eta$-axis is always perpendicular to the meridional section of the body corresponding to the value $\varphi = \pi/2$ and the $G\xi$-axis always lies in the plane of this meridional section (see Fig. 3). Since $G\eta$ is directed along $[\boldsymbol{e}_3\times\boldsymbol{\gamma}]$, we have for the vector $\boldsymbol{e}_\eta$ the following formula:

$$\boldsymbol{e}_\eta = \frac{1}{\sin\theta}[\boldsymbol{e}_3\times\boldsymbol{\gamma}].$$

Taking into account the formulas (3.1), we get

$$\boldsymbol{e}_\eta = -\cos\varphi \boldsymbol{e}_1 + \sin\varphi \boldsymbol{e}_2$$

and, therefore,

$$\boldsymbol{e}_\xi = [\boldsymbol{e}_\eta \times \boldsymbol{e}_\zeta] = [\boldsymbol{e}_\eta \times \boldsymbol{e}_3] = \sin\varphi \boldsymbol{e}_1 + \cos\varphi \boldsymbol{e}_2.$$

Thus, the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are connected with the unit vectors $\boldsymbol{e}_\xi$, $\boldsymbol{e}_\eta$, $\boldsymbol{e}_\zeta$ by the formulas

$$\boldsymbol{e}_1 = \sin\varphi \boldsymbol{e}_\xi - \cos\varphi \boldsymbol{e}_\eta, \quad \boldsymbol{e}_2 = \cos\varphi \boldsymbol{e}_\xi + \sin\varphi \boldsymbol{e}_\eta, \quad \boldsymbol{e}_3 = \boldsymbol{e}_\zeta.$$

The unit normal vector $\boldsymbol{\gamma}$ has the following decomposition in the coordinate system $G\xi\eta\zeta$:

$$\boldsymbol{\gamma} = \sin\theta \boldsymbol{e}_\xi + \cos\theta \boldsymbol{e}_\zeta,$$

and the radius-vector $\overrightarrow{GM}$ of the point M of contact of the body with the horizontal plane has the form

$$\overrightarrow{GM} = (x_1 \sin\varphi + x_2 \cos\varphi)\boldsymbol{e}_\xi + (x_2 \sin\varphi - x_1 \cos\varphi)\boldsymbol{e}_\eta + x_3 \boldsymbol{e}_\zeta = \xi \boldsymbol{e}_\xi + \eta \boldsymbol{e}_\eta + \zeta \boldsymbol{e}_\zeta.$$

Taking into account (3.6) and (3.7), we conclude that

$$\xi = -\delta, \quad \eta = 0, \quad \zeta = x_3,$$

i.e., the components of the vector $\overrightarrow{GM}$ in the coordinate system $G\xi\eta\zeta$ are function only of θ. Hence, the distance from the center of mass of the body to the horizontal supporting plane is also a function only of θ:

$$GQ = -(\overrightarrow{GM} \cdot \boldsymbol{\gamma}) = -\xi \sin\theta - \zeta \cos\theta = f(\theta). \tag{3.8}$$

The equations $\xi = \xi(\theta)$ and $\zeta = \zeta(\theta)$ are the parametric equations of the meridional section of the body (see Fig. 3). Since the vector $\overrightarrow{MQ}$ is a tangent vector to this meridional section at M, it is collinear to the vector whose components in the coordinate system $G\xi\eta\zeta$ are ξ' and ζ', where $(\cdot)'$ denotes the derivative by θ. On the other hand, the vector $\overrightarrow{GQ}$ is collinear to the vector $\boldsymbol{\gamma}$. This means that $\left(\overrightarrow{MQ} \cdot \overrightarrow{GQ}\right) = 0$, i.e.,

$$\xi' \sin\theta + \zeta' \cos\theta = 0. \tag{3.9}$$

Differentiating both sides of (3.8) by θ and applying (3.9) we obtain

$$f'(\theta) = -\xi \cos\theta + \zeta \sin\theta. \tag{3.10}$$

From (3.8) and (3.10) we obtain the parametric equations of the meridional section of the body in the form

$$\xi = -f(\theta)\sin\theta - f'(\theta)\cos\theta, \quad \zeta = -f(\theta)\cos\theta + f'(\theta)\sin\theta. \tag{3.11}$$

Thus, the function $f(\theta)$ completely characterizes the shape of the moving body. In the sequel, we will consider the motion on the body in the coordinate system $G\xi\eta\zeta$.

3.1.2. Equations of motion. We obtain the equations of motion of the rotationally symmetric body on a fixed, perfectly rough horizontal plane from the basic theorems of dynamics. The position of the body on the plane is completely determined by the angles θ, ψ, and φ and by the coordinates x and y of the contact point M. Let the velocity $\boldsymbol{v}$ of the center of mass G, the angular velocity vector $\boldsymbol{\omega}$ of the body, the angular velocity vector $\boldsymbol{\Omega}$ of the coordinate system $G\xi\eta\zeta$, and the reaction $\boldsymbol{R}$ of the plane be specified in the coordinate system $G\xi\eta\zeta$ by their components v_ξ, v_η, v_ζ; p, q, r; Ω_ξ, Ω_η, Ω_ζ, and R_ξ, R_η, R_ζ, respectively. Let m be mass of the body, A_1 be its moment of inertia about axes $G\xi$ and $G\eta$, and A_3 be its moment of inertia about the symmetry axis $G\zeta$. The equations of motion of the body in the coordinate system $G\xi\eta\zeta$ have the following form:

$$m\dot{\boldsymbol{v}} + m[\boldsymbol{\Omega} \times \boldsymbol{v}] = -mg\boldsymbol{\gamma} + \boldsymbol{R}, \tag{3.12}$$

$$\dot{\boldsymbol{K}} + [\boldsymbol{\Omega} \times \boldsymbol{K}] = [\overrightarrow{GM} \times \boldsymbol{R}], \tag{3.13}$$

$$\dot{\boldsymbol{\gamma}} + [\boldsymbol{\Omega} \times \boldsymbol{\gamma}] = \mathbf{0}, \tag{3.14}$$

$$\boldsymbol{v} + [\boldsymbol{\omega} \times \overrightarrow{GM}] = \mathbf{0}. \tag{3.15}$$

Equations (3.12) and (3.13) follow from the theorems of change of momentum and angular momentum of the body, and Eqs. (3.14) and (3.15) express respectively the facts that the vector $\boldsymbol{\gamma}$ is constant in the inertial frame $Oxyz$ and that the body moves without sliding. Here g is the acceleration of gravity and $\boldsymbol{K}$ is the angular momentum of the body with respect to its center of mass. Let ξ, η, ζ be the coordinates of the contact point M in the moving coordinate system $G\xi\eta\zeta$. Then $\eta = 0$, whereas ξ and ζ are defined by (3.11). We can write Eqs. (3.12), (3.13), and (3.15) in the scalar form:

$$\begin{cases} \dfrac{dv_\xi}{dt} + \Omega_\eta v_\zeta - \Omega_\zeta v_\eta = -g\sin\theta + \dfrac{R_\xi}{m}, \\ \dfrac{dv_\eta}{dt} + \Omega_\zeta v_\xi - \Omega_\xi v_\zeta = \dfrac{R_\eta}{m}, \\ \dfrac{dv_\zeta}{dt} + \Omega_\xi v_\eta - \Omega_\eta v_\xi = -g\cos\theta + \dfrac{R_\zeta}{m}; \end{cases} \tag{3.16}$$

$$\begin{cases} A_1\dfrac{dp}{dt} + A_3 r\Omega_\eta - A_1 q\Omega_\zeta = -\zeta R_\eta, \\ A_1\dfrac{dq}{dt} + A_1 p\Omega_\zeta - A_3 r\Omega_\xi = \zeta R_\xi - \xi R_\zeta, \\ A_3\dfrac{dr}{dt} + A_1 q\Omega_\xi - A_1 p\Omega_\eta = \xi R_\eta; \end{cases} \tag{3.17}$$

$$v_\xi + q\zeta = 0, \quad v_\eta + r\xi - p\zeta = 0, \quad v_\zeta - q\xi = 0. \tag{3.18}$$

Now we find the relations between the components of the angular velocity $\boldsymbol{\Omega}$ of the moving trihedron $G\xi\eta\zeta$ and the angular velocity $\boldsymbol{\omega}$ of the body. Since the $G\zeta$-axis is immovable in the body, we have

$$\Omega_\xi = p, \quad \Omega_\eta = q. \tag{3.19}$$

The third component Ω_ζ can be easily expressed through p; indeed, since (due to (3.14)) the vector $\boldsymbol{\Omega}$ os orthogonal to the vector $\dot{\boldsymbol{\gamma}}$, this and (3.19) imply that

$$\Omega_\zeta = \Omega_\xi \cot\theta = p\cot\theta. \tag{3.20}$$

Eliminating the values R_ξ, R_η, and R_ζ from Eqs. (3.16) and (3.17), after certain simplifications based on (3.11) and (3.18)–(3.20), we obtain the equations

$$\begin{gathered} \Big[A_1 + m(\xi^2 + \zeta^2)\Big]\frac{dq}{dt} = mgf'(\theta) + (A_3 r - A_1 p\cot\theta)p \\ - mp(\zeta\cot\theta + \xi)(p\zeta - r\xi) - mq\left(\xi\frac{d\xi}{dt} + \zeta\frac{d\zeta}{dt}\right), \\ A_1\frac{dp}{dt} + A_3\frac{\zeta}{\xi}\frac{dr}{dt} = (A_1 p\cot\theta - A_3 r)q, \quad \frac{d}{dt}(p\zeta - r\xi) - \frac{A_3}{m\xi}\frac{dr}{dt} = (\zeta\cot\theta + \xi)pq. \end{gathered} \tag{3.21}$$

Here ξ and ζ are functions of θ determined by (3.11). Adding to (3.21) the obvious equation

$$q = -\frac{d\theta}{dt}, \tag{3.22}$$

we obtain a closed system of four differential equations with four unknown functions p, q, r, and θ. The equations of motion (3.21)–(3.22) possess the energy integral

$$E = T + V = \text{const}.$$

Using König's theorem and the conditions of the absence of sliding (3.18), we can rewrite this integral as follows:

$$\frac{1}{2}A_1p^2+\frac{1}{2}\bigl(A_1+m(\xi^2+\zeta^2)\bigr)q^2+\frac{1}{2}A_3r^2+\frac{1}{2}m(p\zeta-r\xi)^2+mgf(\theta)=\text{const}\,. \tag{3.23}$$

Assume that $\theta\neq\text{const}$. Then using (3.22) we can introduce the new independent variable θ in the second and third equation of the system (3.21). We obtain

$$\begin{cases} A_1\dfrac{dp}{d\theta}+A_3\dfrac{\zeta}{\xi}\dfrac{dr}{d\theta}=-A_1p\cot\theta+A_3r,\\ \zeta\dfrac{dp}{d\theta}-\dfrac{A_3+m\xi^2}{m\xi}\dfrac{dr}{d\theta}=-\bigl(\zeta\cot\theta+\xi+\zeta'\bigr)p+\xi'r. \end{cases} \tag{3.24}$$

From these linear first-order equations we can obtain one second-order linear differential equation for r. Integrating this equation or the system (3.24), we obtain the dependence of p and r on θ with two arbitrary constants; then the problem can be completed by quadratures. Indeed, if we have found p and r, then we can determine q from the energy integral (3.23). The dependence of the angle θ on time is given by the equation

$$dt=-\frac{d\theta}{q}.$$

Recall that φ is the angle between the meridian $M\zeta$ of the body and a certain fixed meridional plane and ψ is the angle between the horizontal tangent MQ of the meridian $M\zeta$ and the immovable axis Ox (see Fig. 3). Then from the kinematic equations

$$\frac{d\varphi}{dt}=r-\Omega_\zeta=r-p\cot\theta,\quad \frac{d\psi}{dt}=p\sin\theta+\Omega_\zeta\cos\theta=\frac{p}{\sin\theta}. \tag{3.25}$$

we find the angles φ and ψ by quadratures. The coordinates x and y of the point M can also be found by quadratures. Indeed, let $d\sigma_1$ and $d\sigma_2$ be respectively the arc elements of the meridian and the parallel at M; $d\sigma_1$ is oriented from M to Q and $d\sigma_2$ is perpendicular to the plane of Fig. 3 along the $G\eta$-axis. It is easy to see that

$$d\sigma_1=\sqrt{\xi'^2+\zeta'^2}d\theta,\quad d\sigma_2=-\xi d\varphi. \tag{3.26}$$

Since the motion occurs without sliding,

$$dx=d\sigma_1\cos\psi+d\sigma_2\sin\psi,\quad dy=d\sigma_1\sin\psi-d\sigma_2\cos\psi. \tag{3.27}$$

From (3.26) and (3.27) we obtain

$$\begin{cases} \dfrac{dx}{dt}=\sqrt{\xi'^2+\zeta'^2}\cos\psi\dfrac{d\theta}{dt}-\xi\sin\psi\dfrac{d\varphi}{dt},\\ \dfrac{dy}{dt}=\sqrt{\xi'^2+\zeta'^2}\sin\psi\dfrac{d\theta}{dt}+\xi\cos\psi\dfrac{d\varphi}{dt}. \end{cases} \tag{3.28}$$

If θ, ψ, and φ have already been found as functions of t, then x and y can be found from (3.28) by quadratures. Thus, the problem on the rolling if a heavy rotational body on a fixed, perfectly rough horizontal plane is equivalent to the solution of the system (3.24). We solve this system with respect to the derivatives $dp/d\theta$ and $dr/d\theta$:

$$\begin{cases} \dfrac{dp}{d\theta}=\left(-\dfrac{\cos\theta}{\sin\theta}-\dfrac{A_3m\zeta(\xi+\zeta')}{\Delta}\right)p+\dfrac{A_3(A_3+m\xi^2+m\xi'\zeta)}{\Delta}r,\\ \dfrac{dr}{d\theta}=\dfrac{A_1m\xi(\xi+\zeta')}{\Delta}p+\dfrac{m\xi(A_3\zeta-A_1\xi')}{\Delta}r. \end{cases} \tag{3.29}$$

Here we introduce the notation

$$\Delta=A_1A_3+A_1m\xi^2+A_3m\zeta^2.$$

Note that if $\xi + \zeta' = 0$, then the second equation of the system (3.29) can be integrated separately. This condition is valid for the rotational body whose shape is defined by the equations

$$f(\theta) = R - d\cos\theta, \quad \xi = -R\sin\theta, \quad \zeta = d - R\cos\theta,$$

i.e., for a dynamically symmetric ball of radius R whose center of mass is located on the axis $G\zeta$ at the distance d from the geometrical center of the ball. The motion of such nonhomogeneous dynamically symmetric ball was completely examined by S. A. Chaplygin (see [5]). In particular, Chaplygin proved that the system (3.24) or (3.29) possesses two first integrals linear in p and r:

$$A_1 p\sin\theta + A_3 r\left(\cos\theta - \frac{d}{R}\right) = j_1 = \text{const},$$

$$r\sqrt{A_1A_3 + mR^2\left(A_1\sin^2\theta + A_3\left(\cos\theta - \frac{d}{R}\right)^2\right)} = j_2 = \text{const},$$

and hence, it can be solved in Liouville functions. Further, we will assume that $\xi + \zeta' \neq 0$. Then it is possible to obtain from the system (3.29) the following second-order linear differential equation:

$$\frac{d^2r}{d\theta^2} + \left[\frac{\cos\theta}{\sin\theta} + \frac{3m(A_1\xi\xi' + A_3\zeta\zeta')}{\Delta} - \frac{\frac{d}{d\theta}(\xi(\xi+\zeta'))}{\xi(\xi+\zeta')}\right]\frac{dr}{d\theta} + \frac{m\xi(\xi+\zeta')}{\Delta\sin\theta}\left[\frac{d}{d\theta}\left(\frac{(A_1\xi' - A_3\zeta)\sin\theta}{\xi+\zeta'}\right) - A_3\sin\theta\right]r = 0. \quad (3.30)$$

The further analysis of the problem consists of integration of the second-order linear differential equation (3.30). In this paper, we consider the motion of various bodies on a horizontal plane; for each body we present the corresponding equation of the form (3.30) and, using the Kovacic algorithm, examine whether the obtained second-order linear differential equation has a Liouville solution.

3.2. Motion of a thin disk.

3.2.1. Equations of motion. Integrability in terms of hypergeometric functions. First, we consider the problem of motion of a thin disk, which is a rotationally symmetric body with sharp edge, rolling on a horizontal plane. The edge of the disk is a planar circle of radius R centered at the center of mass G. The axis of symmetry of the disk is perpendicular to the plane of the sharp edge. We assume that during the motion the lowest point of the edge remains in contact with the horizontal plane (see [1, 5, 11, 19, 20]). In this case, the formulas (3.8) and (3.11) yield

$$f(\theta) = R\sin\theta, \quad \xi = -R, \quad \zeta = 0. \quad (3.31)$$

Taking into account (3.31), we can write the system (3.24) as follows:

$$(A_3 + mR^2)\frac{dr}{d\theta} = mR^2 p, \quad A_1\frac{d}{d\theta}(p\sin\theta) = A_3 r\sin\theta, \quad (3.32)$$

and the second-order linear differential equation (3.30) takes the form

$$\frac{d^2r}{d\theta^2} + \cot\theta\frac{dr}{d\theta} - Br = 0, \quad B = \frac{mR^2A_3}{A_1(A_3 + mR^2)}, \quad \theta \in (0, \pi). \quad (3.33)$$

Introducing instead of θ the new independent variable z by the formula

$$\cos\theta = 1 - 2z$$

(see [1, 5, 19, 20]), we reduce (3.33) to the form

$$z(1-z)\frac{d^2r}{dz^2} + (1-2z)\frac{dr}{dz} - Br = 0. \quad (3.34)$$

The obtained second-order linear differential equation is the Gauss hypergeometric equation (see [13]). Thus, the problem of motion of a thin round disk is solved in terms of hypergeometric functions. The Gauss hypergeometric series

$$F(\alpha,\beta,\gamma;z) = 1 + \frac{\alpha\beta}{1\cdot\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}z^2 + \dots + \frac{\alpha(\alpha+1)\dots(\alpha+n)\beta(\beta+1)\dots(\beta+n)}{1\cdot 2\dots(n+1)\gamma(\gamma+1)\dots(\gamma+n)}z^{n+1} + \dots, \quad (3.35)$$

is a particular solution of the differential equation

$$z(1-z)\frac{d^2r}{dz^2} + \big[\gamma - (\alpha+\beta+1)z\big]\frac{dr}{dz} - \alpha\beta r = 0.$$

For Eq. (3.34) we have

$$\alpha+\beta = 1, \quad \gamma = 1, \quad \alpha\beta = B = \frac{mR^2A_3}{A_1(A_3+mR^2)}, \qquad \frac{\alpha+\beta+1}{2} = \gamma,$$

and hence its general solution has the form

$$r = c_1F(\alpha,\beta,1;z) + c_2F(\alpha,\beta,1;1-z)$$

(see [13]), where c_1 and c_2 are arbitrary constants and α and β are the roots of the quadratic equation

$$s^2 - s + B = 0.$$

Note (see [13]) that the hypergeometric series (3.35) converges uniformly on any segment of the real axis lying inside the interval $-1 < z < 1$. Turning from z to the old independent variable θ, we obtain the function $r(\theta)$ in the form

$$r = c_1F\left(\alpha,\beta,1;\sin^2\frac{\theta}{2}\right) + c_2F\left(\alpha,\beta,1;\cos^2\frac{\theta}{2}\right). \quad (3.36)$$

From (3.32) and (3.36) we find the function $p = p(\theta)$:

$$p = \frac{A_3}{2A_1}\sin\theta\left[c_1F\left(\alpha+1,\beta+1,2;\sin^2\frac{\theta}{2}\right) - c_2F\left(\alpha+1,\beta+1,2;\cos^2\frac{\theta}{2}\right)\right] \quad (3.37)$$

taking into account the expression for the derivative of the Gauss hypergeometric function

$$\frac{d}{dz}F(\alpha,\beta,\gamma;z) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;z).$$

Thus, the general solution of the system (3.32) can be written in the form (3.36), (3.37) in terms of hypergeometric functions. Hence, in general case, Eq. (3.33) has no Liouville solutions. We try to examine whether can Liouville solutions exist for a certain particular values of the parameter B. For this purpose, we apply the Kovacic algorithm.

3.2.2. Application of the Kovacic algorithm to the problem of motion of a disk. In order to apply the Kovacic algorithm to Eq. (3.33), we first transform its coefficients to the rational form. For this purpose, in Eq. (3.33) we change the independent variable θ by the formula $\cos\theta = x$. As a result, Eq. (3.33) takes the form

$$\frac{d^2r}{dx^2} - \frac{2x}{1-x^2}\frac{dr}{dx} - \frac{B}{1-x^2}r = 0. \quad (3.38)$$

This equation is the starting equation for application of the Kovacic algorithm. If we denote

$$a(x) = -\frac{2x}{1-x^2}, \quad b(x) = -\frac{B}{1-x^2},$$

then Eq. (3.38) has the form (2.1). By the change of variables (2.2), we reduce Eq. (3.38) to the form

$$\frac{d^2y}{dx^2} = D(x)y, \tag{3.39}$$

where

$$D(x) = \frac{2B-1}{4(x+1)} - \frac{2B-1}{4(x-1)} - \frac{1}{4(x+1)^2} - \frac{1}{4(x-1)^2}.$$

The Laurent expansion of $D(x)$ in a neighborhood of the point $x = \infty$ is

$$D(x)\big|_{x=\infty} = -\frac{B}{x^2} + O(\frac{1}{x^4}).$$

All initial preparations necessary for the application of the Kovacic algorithm have been performed.

Remark 1. Here and in the sequel, we consider only values of parameters of the problem that have a physical sense. In other words, we assume that all geometric parameters of the problem are positive, the mass of the body is positive, and the moments of inertia of the body are positive and satisfy the triangle inequality. However, the Kovacic algorithm allows one to find Liouville solutions of a second-order linear differential equation for all values of parameters. For example, if $B = 0$ (i.e., the mass of the disk is concentrated on its symmetry axis), then Eq. (3.33) has a solution expressed through elementary functions:

$$r(\theta) = c_1 \ln\left(\tan\frac{\theta}{2}\right) + c_2. \tag{3.40}$$

Direct application of the Kovacic algorithm to Eq. (3.39) yields the following result.

Theorem 5. *For all physically valid values of parameters, Eq.* (3.39) *has no Liouville solutions.*

Proof. First, we try to search for a solution of Eq. (3.39) that has the form (2.4), i.e., a solution described in Case 1 of Theorem 1. Note that the function $D(x)$ has two finite second-order poles at $x = 1$ and $x = -1$. The point $x = \infty$ is also a second-order pole of $D(x)$. Therefore, the necessary conditions (see Theorem 1) of the existence of a solution to Eq. (3.39) of the form (2.4) are fulfilled. Now we apply the Kovacic algorithm following Sec. 2.3.1.

Step 1. Let us calculate the following values:

$$[\sqrt{D}]_{\pm 1} = 0, \qquad b_{\pm 1} = -\frac{1}{4}, \quad \alpha_1^{\pm} = \alpha_{-1}^{\pm} = \frac{1}{2},$$
$$[\sqrt{D}]_{\infty} = 0, \qquad b_{\infty} = -B, \quad \alpha_{\infty}^{\pm} = \frac{1 \pm \sqrt{1-4B}}{2}.$$

Step 2. Since the number ρ of finite poles of the function $D(x)$ is equal to 2, we have $2^{\rho+1} = 2^3 = 8$ tuples of signs $s = (s(\infty), s(1), s(-1))$. For each tuple, we calculate d by the formula (2.19):

$$d = \alpha_{\infty}^{s(\infty)} - \alpha_1^{s(1)} - \alpha_{-1}^{s(-1)}.$$

It is easy to verify that only two values of d are distinct, namely,

$$d = \frac{\pm\sqrt{1-4B} - 1}{2}.$$

According to the algorithm, d must be nonnegative integer. However, according to the physical sense of parameters, we have $B > 0$. Therefore, in this case, d can only be negative. Hence, Eq. (3.39) has no Liouville solutions of the form (2.4). Note that for $B = 0$, we have the unique nonnegative integer $d = 0$ and Eq. (3.39) possesses a Liouville solution, which has the form (3.40) for Eq. (3.33).

Now we try to search for a solution of the form (2.9) for Eq. (3.39), i.e., a solution corresponding to Case 2 of Theorem 1. The necessary conditions for the existence of such a solution are fulfilled (see Theorem 4). We apply the Kovacic algorithm as was described in Sec. 2.3.3.

Step 1. Let us define the following sets of integers:

$$E_1 = \{2\}, \quad E_{-1} = \{2\}, \quad E_\infty = \Big\{(2 \pm k\sqrt{1-4B}) \cap \mathbb{Z},\ k = 0, \pm 2\Big\}.$$

It is easy to see that either $E_\infty = \{1, 2, 3\}$ for $B = 3/16$ or $E_\infty = \{2\}$ for any other admissible values of B satisfying the conditions $B \geq 0$ and $1 - 4B \geq 0$.

Step 2. Now we consider all possible tuples $s = (e_\infty, e_1, e_{-1})$ of elements of the sets E_∞, E_1, and E_{-1}; at least one of the elements in each set must be odd. Obviously, odd numbers in the set s can appear only for $B = 3/16$. We have exactly two such sets: $(1, 2, 2)$ and $(3, 2, 2)$. However, the corresponding values of d calculated by the formula (2.28)

$$d = \frac{1}{2}(e_\infty - e_1 - e_{-1}),$$

are not nonnegative integers: we have $d = -3/2$ for the set $(1, 2, 2)$ and $d = -1/2$ for the set $(3, 2, 2)$. Therefore Eq. (3.39) has no Liouville solutions of the form (2.9).

Finally, we try to search for a solution of the form (2.13) to Eq. (3.39), i.e., a solution corresponding to Case 3 of Theorem 1. First, we analyze the necessary conditions of the existence (see Theorem 4). The function $D(x)$ has no poles of order greater than 2. The order of the pole of $D(x)$ at $x = \infty$ is greater than 1. The partial fraction expansion of $D(x)$ is

$$D(x) = \frac{\alpha_1}{(x+1)^2} + \frac{\alpha_2}{(x-1)^2} + \frac{\beta_1}{(x+1)} + \frac{\beta_2}{(x-1)},$$
$$\alpha_1 = -\frac{1}{4}, \quad \alpha_2 = -\frac{1}{4}, \quad \beta_1 = \frac{2B-1}{4}, \quad \beta_2 = -\frac{2B-1}{4},$$

It can be easily shown that

$$\sqrt{1+4\alpha_1} = 0 \in \mathbb{Q}, \quad \sqrt{1+4\alpha_2} = 0 \in \mathbb{Q}, \quad \beta_1 + \beta_2 = 0.$$

Thus, for the existence of a solution (2.13) to Eq. (3.39), the following condition must be fulfilled:

$$\sqrt{1+4\gamma} \in \mathbb{Q}, \quad \gamma = \alpha_1 + \alpha_2 - \beta_1 + \beta_2.$$

For Eq. (3.39), this condition can written in the form

$$\sqrt{1-4B} \in \mathbb{Q}.$$

Assume that this condition holds. Now we apply the Kovacic algorithm for Case 3 as described in Sec. 2.3.5.

Step 1. Let us define the following sets of integers:

$$E_1 = \{6\}, \quad E_{-1} = \{6\}, \quad E_\infty = \Big\{(6 + k\sqrt{1-4B}) \cap \mathbb{Z},\ k = 0,\ \pm 1,\ \pm 2,\ \ldots, \pm 6\Big\}.$$

Step 2. Consider all possible tuples $s = (e_\infty, e_1, e_{-1})$ of elements of the sets E_∞, E_1, and E_{-1} and calculate d by the formula (2.41):

$$d = e_\infty - e_1 - e_{-1} = k\sqrt{1-4B} - 6.$$

According to the algorithm, d must be nonnegative integer. Note that $k = 6$ is the maximal possible value for k. Therefore, d cannot be a nonnegative integer for $B > 0$. This means that Eq. (3.39) does not possess a solution of the form (2.13). Thus, we can conclude that Eq. (3.39) has no Liouville solutions for $B > 0$. The theorem is proved. □

Thus, we have proved that the problem of motion of a round disk on a perfectly rough plane cannot be solved in terms of Liouville functions. In other words, the hypergeometric functions (3.36), (3.37) cannot be reduced to any simpler functions for all physically admissible values of the parameter B.

3.3. Motion of a thick disk. Mow we consider the problem of the motion of a rotationally symmetric disk whose sharp edge (a planar circle of radius R) rolls on a horizontal plane. The symmetry axis $G\zeta$ of the body is perpendicular to the plane of the circle and passes through its center. The center of mass G of the body is also situated on the $G\zeta$-axis, but in the case considered, it does not coincide with the center of the circle. Let h be the distance between the center of mass of the body and the center of the circle. Such a body is called a *disk of finite thickness* or a *thick disk* (see [2–4, 30]). The distance between the center of mass of the disk and the horizontal supporting plane is

$$f(\theta) = R\sin\theta + h\cos\theta.$$

According to (3.11) we have

$$\xi = -R, \quad \zeta = -h.$$

The system (3.29) takes the form

$$\begin{cases} \dfrac{dp}{d\theta} = -\Big(\dfrac{\cos\theta}{\sin\theta} + \dfrac{A_3mRh}{A_1A_3 + A_1mR^2 + A_3mh^2}\Big)p + \dfrac{A_3(A_3+mR^2)}{A_1A_3 + A_1mR^2 + A_3mh^2}r, \\ \dfrac{dr}{d\theta} = \dfrac{A_1mR^2}{A_1A_3 + A_1mR^2 + A_3mh^2}p + \dfrac{A_3mRh}{A_1A_3 + A_1mR^2 + A_3mh^2}r, \end{cases} \tag{3.41}$$

and Eq. (3.30) can be written as follows:

$$\frac{d^2r}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dr}{d\theta} - \frac{A_3mR(R\sin\theta + h\cos\theta)}{(A_1A_3 + A_1mR^2 + A_3mh^2)\sin\theta}r = 0, \quad \theta \in \left(0,\ \pi - \arccos\frac{R}{\sqrt{R^2+h^2}}\right). \tag{3.42}$$

The general solution of the system (3.41) expressed in terms of hypergeometric functions was first found by M. Batista (see [2–4]). In the problem considered, explicit expressions for $p(\theta)$ and $r(\theta)$ are more complicated than the corresponding expressions (3.36) and (3.37) in the problem of the motion of a thin disk. We do not describe in detail the procedure of calculating the general solution of the system (3.41) or Eq.(3.42). Let us examine the question on the existence of Liouville solutions of Eq. (3.42). In this differential equation, we perform the change of the independent variable by the formula $\cot\theta = x$ and introduce the following notation:

$$A = \frac{A_3mhR}{A_1A_3 + A_1mR^2 + A_3mh^2}, \quad B = \frac{A_3mR^2}{A_1A_3 + A_1mR^2 + A_3mh^2}. \tag{3.43}$$

Then Eq. (3.42) becomes

$$\frac{d^2r}{dx^2} + \frac{x}{x^2+1}\frac{dr}{dx} - \frac{Ax+B}{(x^2+1)^2}r = 0. \tag{3.44}$$

Let

$$a(x) = \frac{x}{x^2+1}, \quad b(x) = -\frac{Ax+B}{(x^2+1)^2};$$

then Eq. (3.44) takes the form of Eq. (2.1). By the change of variables (2.2), we reduce Eq. (3.44) to the form

$$\frac{d^2y}{dx^2} = D_1(x)y, \quad D_1(x) = \frac{(4B+1)i}{16(x+i)} - \frac{3+4B-4Ai}{16(x+i)^2} - \frac{(4B+1)i}{16(x-i)} - \frac{3+4B+4Ai}{16(x-i)^2}. \tag{3.45}$$

The Laurent expansion of $D_1(x)$ in a neighborhood of the point $x = \infty$ is

$$D_1(x)\big|_{x=\infty} = -\frac{1}{4x^2} + O\left(\frac{1}{x^3}\right).$$

All initial preparations necessary for application of the Kovacic algorithm to Eq. (3.45) are fulfilled. Direct application of the Kovacic algorithm to Eq. (3.45) yields the following result.

Theorem 6. *For all physically admissible values of the parameters of the problem, Eq.* (3.45) *has no Liouville solutions.*

Proof. First, for Eq. (3.45), we try to search for a solution of the form (2.4), i.e., a solution described in Case 1 of Theorem 1. Note that the function $D_1(x)$ has two finite second-order poles at $x = i$ and $x = -i$. The order of the pole of $D_1(x)$ at $x = \infty$ is also 2. Therefore, the conditions of Theorem 4 necessary for the existence of solutions of the form (2.4) for Eq. (3.45) are fulfilled. Now we start to apply the Kovacic algorithm as was described in Sec. 2.3.1.

Step 1. Let us calculate the following values:

$$[\sqrt{D_1}]_{\pm i} = 0, \quad \alpha_i^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1}{4} - B - Ai}, \quad \alpha_{-i}^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1}{4} - B + Ai},$$
$$[\sqrt{D_1}]_{\infty} = 0, \quad b_\infty = -\frac{1}{4}, \quad \alpha_\infty^{\pm} = \frac{1}{2}.$$

Step 2. Since the number ρ of finite poles of the function $D_1(x)$ is equal to 2, we have $2^{\rho+1} = 2^3 = 8$ tuples of signs $s = (s(\infty), s(i), s(-i))$. For each tuple, we evaluate the value d by (2.19):

$$d = \alpha_\infty^{s(\infty)} - \alpha_i^{s(i)} - \alpha_{-i}^{s(-i)}.$$

As a result, we have the following explicit expression for d:

$$d = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{4} - B - Ai} + \frac{1}{2}\sqrt{\frac{1}{4} - B + Ai}.$$

Here the radical means a pair of two opposite complex numbers. After calculation of square roots, we obtain the four numbers $\pm u \pm vi$, where

$$u = \left|\operatorname{Re}\sqrt{\frac{1}{4} - B + Ai}\right|, \quad v = \left|\operatorname{Im}\sqrt{\frac{1}{4} - B + Ai}\right|.$$

Therefore, d can be a nonnegative integer only if

$$d = -\frac{1}{2} + \frac{1}{2}(u + vi + u - vi) = -\frac{1}{2} + \left|\operatorname{Re}\sqrt{\frac{1}{4} - B + Ai}\right|.$$

It is easy to verify that

$$d = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1 - 4B + \sqrt{16A^2 + (1 - 4B)^2}}{2}}. \tag{3.46}$$

We prove that if A and B are defined by (3.43), then the value d defined by (3.46) cannot be nonnegative integer. We rewrite Eq. (3.46) as follows:

$$16A^2 = 4(2d + 1)^4 - 4(2d + 1)^2(1 - 4B).$$

Introduce the notation $2(2d + 1)^2 = k \geq 2$ for $d \geq 0$. Taking into account this notation, we obtain

$$16A^2 = k^2 - 2k(1 - 4B)$$

or

$$8B\left(2\frac{A^2}{B} - k\right) = k(k - 2). \tag{3.47}$$

The right-hand side of Eq. (3.47) is nonnegative for $k \geq 2$, whereas the left-hand side is negative. Indeed, from (3.43) we get

$$\frac{A^2}{B} = \frac{A_3 m h^2}{A_1 A_3 + A_1 m R^2 + A_3 m h^2} < \frac{A_3 m h^2}{A_3 m h^2} = 1$$

(recall that $B > 0$). The contradiction obtained allows one to state that Eq. (3.45) has no Liouville solutions of the form (2.4).

Now we search for a solution of the form (2.9) for Eq. (3.45), i.e., a solution described in Case 2 of Theorem 1. The necessary conditions for the existence of such solutions are fulfilled (see Theorem 4). Now we apply the Kovacic algorithm as was described in Sec. 2.3.3.

Step 1. Let us define the following sets of integers:

$$E_{\pm i} = \left\{ \left(2 + k\sqrt{\frac{1}{4} - B \mp Ai} \right) \cap \mathbb{Z} \right\} = \{2\}, \quad k = 0, \pm 2, \qquad E_\infty = \{2\}.$$

Step 2. Now we should consider all possible tuples $s = (e_\infty, e_i, e_{-i})$ of elements from E_∞, E_i, E_{-i}; recall that at least one element in each tuple must be odd. Obviously, in the case considered, there are no such sets. Thus, Eq. (3.45) has no Liouville solutions of the form (2.9).

Now we search for a solution (2.13) of Eq. (3.45), i.e., a solution described in Case 3 of Theorem 1. First, we verify whether the necessary conditions for its existence hold (see Theorem 4). The function $D_1(x)$ can be written as follows:

$$D_1(x) = \frac{\alpha_1}{(x+i)^2} + \frac{\alpha_2}{(x-i)^2} + \frac{\beta_1}{x+i} + \frac{\beta_2}{x-i},$$

where

$$\alpha_1 = -\frac{3 + 4B - 4Ai}{16}, \quad \alpha_2 = -\frac{3 + 4B + 4Ai}{16}, \quad \beta_1 = \frac{(4B+1)i}{16}, \quad \beta_2 = -\frac{(4B+1)i}{16}.$$

Therefore, we have

$$\sqrt{1 + 4\alpha_1} = \sqrt{\frac{1}{4} - B + Ai} \notin \mathbb{Q}, \quad \sqrt{1 + 4\alpha_2} = \sqrt{\frac{1}{4} - B - Ai} \notin \mathbb{Q}.$$

Thus, the necessary conditions for the existence of a solution of the form (2.13) for Eq. (3.45) are not fulfilled. This means that Eq. (3.45) has no Liouville solutions of the form (2.13). Finally, we can conclude that Eq. (3.45) has no Liouville solutions. The theorem is proved. ☐

Thus, we have proved that the problem of the motion of a thick disk on a perfectly rough horizontal plane cannot be solved in terms of Liouville functions.

4. Motion of Torus

4.1. Formulation of the problem. Equations of motion. General case and special cases. Consider the problem of the motion of a dynamically symmetric torus on a perfectly rough horizontal plane. Let R be the radius of the meridian of the torus (called the radius of the tube) and a be the distance from the center of the tube to the center of the torus. We assume that $a > R$. The center of mass of the torus is located at its center (see Fig. 4). Then the distance between the the center of mass and the horizontal supporting plane is

$$f(\theta) = R + a \sin\theta.$$

According to (3.11) we have

$$\xi = -a - R\sin\theta, \quad \zeta = -R\cos\theta.$$

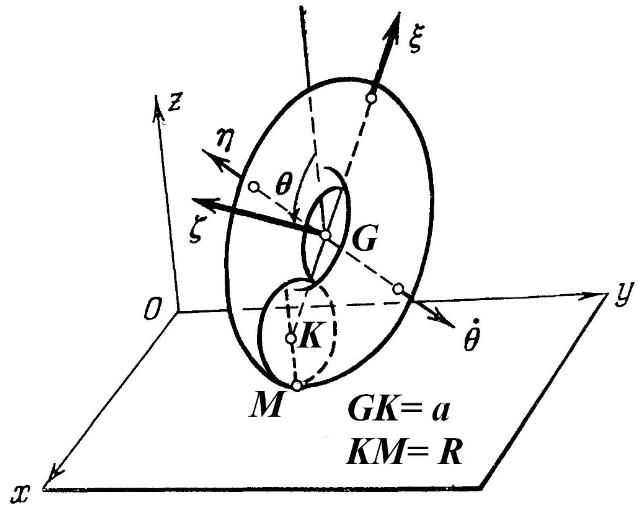


Fig. 4. Torus rolling on a horizontal plane.

Therefore, the system (3.29) can be written as follows:

$$\begin{cases} \dfrac{dp}{d\theta} = -\left(1 + \dfrac{A_3maR\sin\theta}{\Delta}\right)\dfrac{\cos\theta}{\sin\theta}p + \dfrac{A_3(A_3 + ma^2 + mR^2 + 2maR\sin\theta)}{\Delta}r, \\ \dfrac{dr}{d\theta} = \dfrac{A_1ma(R\sin\theta + a)}{\Delta}p + \dfrac{mR(A_3 - A_1)(R\sin\theta + a)\cos\theta}{\Delta}r, \\ \Delta = (A_1 - A_3)mR^2\sin^2\theta + 2A_1mRa\sin\theta + A_1A_3 + A_1ma^2 + A_3mR^2, \end{cases} \tag{4.1}$$

and the differential equation (3.30) takes the form

$$\frac{d^2r}{d\theta^2} + b_1\frac{dr}{d\theta} + b_2r = 0, \quad \theta \in (0, \pi), \tag{4.2}$$

where

$$\begin{aligned} b_1 &= \frac{a\cos\theta}{(R\sin\theta + a)\sin\theta} + \frac{3mR((A_1 - A_3)R\sin\theta + A_1a)\cos\theta}{\Delta}, \\ b_2 &= \frac{m(R\sin\theta + a)(R(A_1 - A_3)(1 - 2\sin^2\theta) - A_3a\sin\theta)}{\Delta\sin\theta}, \\ \Delta &= (A_1 - A_3)mR^2\sin^2\theta + 2A_1mRa\sin\theta + A_1A_3 + A_1ma^2 + A_3mR^2. \end{aligned}$$

The system (4.1) was first obtained by L. G. Lobas (see [28, 29, 36]). We examine the question on the existence of Liouville solutions of Eq. (4.2). We perform in Eq. (4.2) the change of the θ by the formula $\sin\theta = x$ and introduce the notation $B = a/R$. Since $a > R$ by assumption, we have $B > 1$. Thus, we rewrite (4.2) as follows:

$$\frac{d^2r}{dx^2} + d_1(x)\frac{dr}{dx} + d_2(x)r = 0, \tag{4.3}$$

where

$$\begin{aligned} d_1(x) &= \frac{B}{x(x+B)} + \frac{x}{x^2 - 1} + \frac{3mR^2((A_1 - A_3)x + A_1B)}{\Delta}, \\ d_2(x) &= \frac{mR^2(x+B)((A_1 - A_3)(2x^2 - 1) + A_3Bx)}{x(x^2 - 1)\Delta}, \\ \Delta &= (A_1 - A_3)mR^2x^2 + 2A_1BmR^2x + A_1A_3 + A_1B^2mR^2 + A_3mR^2. \end{aligned}$$

If $A_3 \neq A_1$, then the polynomial Δ has two roots x_1 and x_2:

$$
\begin{aligned}
x_1 &= -\frac{A_1 mRB - \sqrt{A_3 m(A_1 mR^2B^2 - (A_1 - A_3)(A_1 + mR^2))}}{mR(A_1 - A_3)}, \\
x_2 &= -\frac{A_1 mRB + \sqrt{A_3 m(A_1 mR^2B^2 - (A_1 - A_3)(A_1 + mR^2))}}{mR(A_1 - A_3)}.
\end{aligned}
\tag{4.4}
$$

The change of variable (2.2) reduces Eq. (4.3) to the form

$$
\frac{d^2y}{dx^2} = T(x)y, \tag{4.5}
$$

where

$$
\begin{aligned}
T(x) = \frac{\beta_1}{x-1} + \frac{\alpha_1}{(x-1)^2} + \frac{\beta_2}{x+1} + \frac{\alpha_2}{(x+1)^2} + \frac{\beta_3}{x-x_1} + \frac{\alpha_3}{(x-x_1)^2} \\
+ \frac{\beta_4}{x-x_2} + \frac{\alpha_4}{(x-x_2)^2} + \frac{\beta_5}{x+B} + \frac{\alpha_5}{(x+B)^2} + \frac{\beta_6}{x} + \frac{\alpha_6}{x^2},
\end{aligned}
\tag{4.6}
$$

$$
\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\frac{3}{16}, \quad \alpha_5 = \frac{3}{4}, \quad \alpha_6 = -\frac{1}{4},
$$

$$
\beta_1 = \frac{8B^3 + 4(x_1 + x_2 + 2)B^2 + (5x_1x_2 - 3x_1 - 3x_2 + 9)B}{16(x_1 - 1)(x_2 - 1)(B + 1)} + \frac{x_1x_2 - 3x_1 - 3x_2 + 5}{16(x_1 - 1)(x_2 - 1)(B + 1)},
$$

$$
\beta_2 = -\frac{8B^3 + 4(x_1 + x_2 - 2)B^2 + (5x_1x_2 + 3x_1 + 3x_2 + 9)B}{16(x_1 + 1)(x_2 + 1)(B - 1)} + \frac{x_1x_2 + 3x_1 + 3x_2 + 5}{16(x_1 + 1)(x_2 + 1)(B - 1)},
$$

$$
\begin{aligned}
\beta_3 = \frac{8x_1B^3 + 4(x_1^2 + x_1x_2 + 2)B^2 + (5x_1^3 - 4x_1^2x_2 + x_1 + 6x_2)B}{8x_1(x_1 - 1)(x_1 + 1)(x_1 - x_2)(x_1 + B)} \\
+ \frac{(3x_1^2 - 2x_1x_2 - 1)x_1^2}{8x_1(x_1 - 1)(x_1 + 1)(x_1 - x_2)(x_1 + B)},
\end{aligned}
$$

$$
\begin{aligned}
\beta_4 = \frac{8x_2B^3 + 4(x_2^2 + x_1x_2 + 2)B^2 + (5x_2^3 - 4x_1x_2^2 + 6x_1 + x_2)B}{8x_2(x_2 - 1)(x_2 + 1)(x_2 - x_1)(x_2 + B)} \\
+ \frac{(3x_2^2 - 2x_1x_2 - 1)x_2^2}{8x_2(x_2 - 1)(x_2 + 1)(x_2 - x_1)(x_2 + B)},
\end{aligned}
$$

$$
\beta_5 = \frac{10B^4 + 7(x_1 + x_2)B^3 + 4(x_1x_2 - 2)B^2 - 5(x_1 + x_2)B - 2x_1x_2}{4B(B - 1)(B + 1)(x_1 + B)(x_2 + B)},
$$

$$
\beta_6 = -\frac{4B^2 + 3(x_1 + x_2)B + 2x_1x_2}{4x_1x_2B}.
$$

Thus, the function $T(x)$ has six finite poles: $x = 0$, $x = 1$, $x = -1$, $x = -B$, $x = x_1$, and $x = x_2$. In the general case, these poles are distinct. Nevertheless, under some additional conditions for parameters, the function $T(x)$ has another form different from (4.6). This takes place in the following cases:

1. If

$$
A_3 = A_1, \tag{4.7}
$$

then Δ is a first-degree polynomial and its unique root x_0 is

$$
x_0 = -\frac{A_1 + mR^2 + mR^2B^2}{2mBR^2}. \tag{4.8}
$$

2. For x_0 expressed by (4.8), we have $x_0 < -1$. Hence $x_0 \neq 0, 1, -1$. However, under the condition

$$A_3 = A_1 = mR^2(B^2 - 1) = m(a^2 - R^2) \tag{4.9}$$

we get $x_0 = -B$.

3. If

$$A_3 \neq A_1, \quad A_1 = m(a^2 - R^2), \tag{4.10}$$

then we have $x_1 = -B$.

4. Under the condition

$$A_3 = \frac{A_1}{A_1 + mR^2}(A_1 + mR^2 - ma^2) \tag{4.11}$$

the polynomial Δ has a multiple root:

$$x_1 = x_2 = -\frac{A_1 + mR^2}{mBR^2}.$$

The condition (4.11) has a physical sense if $A_1 + mR^2 - ma^2 > 0$. Then

$$x_1 = x_2 < -B.$$

Consequently, under the condition (4.11), the multiple root $x_1 = x_2$ does not coincide with other poles $x = 0$, $x = 1$, $x = -1$, $x = -B$.

We see that in the examining the existence of Liouville solutions for Eq. (4.5), we must consider the general case where the function $T(x)$ has the form (4.6), and four special cases where the parameters of the problem satisfy one of the relations (4.7), (4.9), (4.10), or (4.11).

4.2. General case. Assume that the function $T(x)$ is determined by formula (4.6), i.e., all its poles are distinct. In this case, the Laurent expansion of $T(x)$ at $x = \infty$ is

$$T(x)\Big|_{x=\infty} = \frac{12B^2 + 4(x_1 + x_2)B + 2x_1x_2 - 3x_1^2 - 3x_2^2 - 8}{16x^4} + O\left(\frac{1}{x^5}\right).$$

Direct application of the Kovacic algorithm to Eq. (4.5) yields the following result.

Theorem 7. *Assume that all poles of the function $T(x)$ are distinct. Then Eq.* (4.5) *has no Liouville solutions for any physically admissible values of parameters.*

Proof. First, we search for a solution of Eq. (4.5) of the form (2.4), i.e., a solution described in Case 1 of Theorem 1. Note that the function $T(x)$ has six finite second-order poles: $x = 0$, $x = 1$, $x = -1$, $x = -B$, $x = x_1$, and $x = x_2$. The order of $T(x)$ at $x = \infty$ is at least fourth. Therefore, the conditions of Theorem 4 (the necessary conditions of the existence of a solution of Eq. (4.5) of the form (2.4)) are fulfilled. Now we apply the Kovacic algorithm as was described in Sec. 2.3.1.

Step 1. Let us calculate the following values:

$$[\sqrt{T}]_1 = 0, \quad \alpha_1^+ = \frac{3}{4}, \quad \alpha_1^- = \frac{1}{4},$$

$$[\sqrt{T}]_{-1} = 0, \quad \alpha_{-1}^+ = \frac{3}{4}, \quad \alpha_{-1}^- = \frac{1}{4},$$

$$[\sqrt{T}]_{x_1} = 0, \quad \alpha_{x_1}^+ = \frac{3}{4}, \quad \alpha_{x_1}^- = \frac{1}{4},$$

$$[\sqrt{T}]_{x_2} = 0, \quad \alpha_{x_2}^+ = \frac{3}{4}, \quad \alpha_{x_2}^- = \frac{1}{4},$$

$$[\sqrt{T}]_{-B} = 0, \quad \alpha_{-B}^+ = \frac{3}{2}, \quad \alpha_{-B}^- = -\frac{1}{2},$$

$$[\sqrt{T}]_0 = 0, \quad \alpha_0^+ = \frac{1}{2}, \quad \alpha_0^- = \frac{1}{2},$$

$$[\sqrt{T}]_\infty = 0, \quad \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1.$$

Step 2. Since the number ρ of finite poles of the function $T(x)$ is equal to 6, we have $2^{\rho+1} = 2^7 = 128$ tuples of signs

$$s = \Big(s(\infty),\ s(1),\ s(-1),\ s(x_1),\ s(x_2),\ s(-B),\ s(0)\Big).$$

For each tuple, we evaluate the value d by the formula (2.19):

$$d = \alpha_\infty^{s(\infty)} - \alpha_1^{s(1)} - \alpha_{-1}^{s(-1)} - \alpha_{x_1}^{s(x_1)} - \alpha_{x_2}^{s(x_2)} - \alpha_{-B}^{s(-B)} - \alpha_0^{s(0)}.$$

According to the algorithm, d must be nonnegative integer. Further, we analyze all possible tuples of signs s and the corresponding values α. It is easy to verify that the unique tuple s such that d is nonnegative is

$$\alpha = \Big(\alpha_\infty^-,\ \alpha_1^-,\ \alpha_{-1}^-,\ \alpha_{x_1}^-,\ \alpha_{x_2}^-,\ \alpha_{-B}^-,\ \alpha_0^-\Big) = \left(1,\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ -\frac{1}{2},\ \frac{1}{2}\right),$$

and $d = 0$. The corresponding function $\theta = \theta(x)$ calculated by the formula (2.20) has the form

$$\theta = \frac{1}{4(x-1)} + \frac{1}{4(x+1)} + \frac{1}{4(x-x_1)} + \frac{1}{4(x-x_2)} - \frac{1}{2(x+B)} + \frac{1}{2x}.$$

Step 3. For the tuple of signs s found in Step 2, we search for a polynomial P of degree $d = 0$ satisfying Eq. (2.21). Since the polynomial P has a zero degree, we set $P \equiv 1$ and substitute it to Eq. (2.21), which takes the form

$$-\frac{B(2B + x_1 + x_2)(Bx + 1)}{2x(x^2-1)(x+B)(x-x_1)(x-x_2)} = 0.$$

Using the explicit expression for x_1 and x_2 (see (4.4)) and the inequality $B > 0$, we conclude that for the existence of a solution of the form (2.4) for Eq. (4.5), the following condition must be valid:

$$2B + x_1 + x_2 = \frac{2A_3B}{A_3 - A_1} = 0.$$

Since $B > 0$ and $A_3 > 0$, the last condition cannot hold. Thus, the differential equation (4.5) has no Liouville solutions of the form (2.4).

Now we try to search for a solution of the form (2.9) of Eq. (4.5), i.e., a solution described in Case 2 of Theorem 1. The necessary conditions for existence of a such solution hold (see Theorem 4). Now we apply the Kovacic algorithm.

Step 1. Let us define the following sets of integers:

$$E_1 = \{1,2,3\},\qquad E_{-1} = \{1,2,3\},\qquad E_{x_1} = \{1,2,3\},\qquad E_{x_2} = \{1,2,3\},$$
$$E_{-B} = \{-2,2,6\},\qquad E_0 = \{2\},\qquad E_\infty = \{0,2,4\}.$$

Step 2. Now we consider all possible tuples

$$s = \big(e_\infty,\ e_1,\ e_{-1},\ e_{x_1},\ e_{x_2},\ e_{-B},\ e_0\big)$$

of elements from E_∞, E_1, E_{-1}, E_{x_1}, E_{x_2}, E_{-B}, E_0, and at least one of the elements in each tuple must be odd. Using (2.28), for each tuple s we get

$$d = \frac{1}{2}\big(e_\infty - e_1 - e_{-1} - e_{x_1} - e_{x_2} - e_{-B} - e_0\big).$$

According to the algorithm, d must be a nonnegative integer. Analyzing all possible tuples s, we conclude that the unique set with nonnegative d is

$$e = \big(e_\infty,\ e_1,\ e_{-1},\ e_{x_1},\ e_{x_2},\ e_{-B},\ e_0\big) = \big(4,\ 1,\ 1,\ 1,\ 1,\ -2,\ 2\big),$$

and $d = 0$.

Step 3. Using (2.29), we form the rational function θ for the chosen tuple e found in Step 2. We get

$$\theta = \frac{1}{2(x-1)} + \frac{1}{2(x+1)} + \frac{1}{2(x-x_1)} + \frac{1}{2(x-x_2)} - \frac{1}{x+B} + \frac{1}{x}.$$

A polynomial of degree $d=0$ ($P \equiv 1$) must satisfy Eq. (2.30). Substituting $P \equiv 1$ in this equation, we obtain

$$\frac{B(2B+x_1+x_2)(Bx^2-(2B^2-2)x-B)}{x^2(x+B)^2(x^2-1)(x-x_1)(x-x_2)} = 0.$$

Since none of the factors can be zero, this equation has no solutions. Thus, Eq. (4.5) has no Liouville solutions of the form (2.9).

Now we search for a solution of the form (2.13) of Eq. (4.5), i.e., a solution described in Case 3 of Theorem 1. First, let us check whether the necessary conditions hold (see Theorem 4). The function $T(x)$ has no poles of order greater than 2. The order of the pole of $T(x)$ at $x=\infty$ is greater than 1. The partial fraction expansion of $T(x)$ is (4.6). It can be easily proved that the remaining conditions of Theorem 4 hold:

$$\sqrt{1+4\alpha_i} = \frac{1}{2} \in \mathbb{Q} \quad (i=1,\dots,4), \qquad \sqrt{1+4\alpha_5} = 2 \in \mathbb{Q}, \quad \sqrt{1+4\alpha_6} = 0 \in \mathbb{Q},$$

$$\sum_{i=1}^{6} \beta_i = 0, \qquad \sqrt{1+4\gamma} = 1 \in \mathbb{Q}, \quad \gamma = 0.$$

Now we start to apply the Kovacic algorithm for Case 3 (see Sec. 2.3.5).

Step 1. Let us define the following sets of integers:

$$E_1 = \{3,4,5,6,7,8,9\}, \qquad E_{-1} = \{3,4,5,6,7,8,9\},$$
$$E_{x_1} = \{3,4,5,6,7,8,9\}, \qquad E_{x_2} = \{3,4,5,6,7,8,9\},$$
$$E_{-B} = \{-6,-4,-2,0,2,4,6,8,10,12,14,16,18\}, \qquad E_0 = \{6\},$$
$$E_\infty = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}.$$

Step 2. Now we consider all possible tuples

$$s = \big(e_\infty,\ e_1,\ e_{-1},\ e_{x_1},\ e_{x_2},\ e_{-B},\ e_0\big)$$

of elements of the sets E_∞, E_1, E_{-1}, E_{x_1}, E_{x_2}, E_{-B}, and E_0. By the formula (2.41), we calculate the number d:

$$d = e_\infty - e_1 - e_{-1} - e_{x_1} - e_{x_2} - e_{-B} - e_0,$$

which must be a nonnegative integer. By analyzing all possible tuples of elements of E_∞, E_1, E_{-1}, E_{x_1}, E_{x_2}, E_{-B}, and E_0, we conclude that the unique tuple with nonnegative d is

$$e = \big(e_\infty,\ e_1,\ e_{-1},\ e_{x_1},\ e_{x_2},\ e_{-B},\ e_0\big) = \big(12,\ 3,\ 3,\ 3,\ 3,\ -6,\ 6\big),$$

and $d=0$.

Step 3. By the formula (2.42), we construct the function θ using the tuple e obtained on Step 2. Then we get

$$\theta = \frac{3}{x-1} + \frac{3}{x+1} + \frac{3}{x-x_1} + \frac{3}{x-x_2} - \frac{6}{x+B} + \frac{6}{x}.$$

Using (2.43), we construct the polynomial

$$S = x(x-1)(x+1)(x-x_1)(x-x_2)(x+B).$$

Further, we need the following recursive formulas (2.44):

$$P_{12} = -P, \quad P_{i-1} = -SP_i' + ((12-i)S' - S\theta)P_i - (12-i)(i+1)S^2T(x)P_{i+1}, \quad P_{-1} = 0,$$

where $P_{12} = -P \equiv -1$ is a polynomial of degree $d = 0$. The numerator of P_{-1} is a polynomial, which must be identically zero. Therefore, all its coefficients must be equal to zero. From this condition one can derive (using a computer algebra system) that either $B = 0$, $x_1 = 0$, $x_2 = 0$ or $x_1 + x_2 + 2B = 0$. It was proved above that $B \neq 0$ and $x_1 + x_2 + 2B \neq 0$. The inequalities $x_1 \neq 0$ and $x_2 \neq 0$ follow from (4.4). This means that Eq. (4.5) has no Liouville solutions of the form (2.13). Finally, we conclude that if all poles of $T(x)$ are distinct, then Eq. (4.5) has no Liouville solutions. The theorem is proved. $\square$

4.3. Special case $A_3 = A_1 \neq m\left(a^2 - R^2\right)$. Now we assume that the moments of inertia of the torus satisfy (4.7). Let $A_3 = A_1 = A$. Then Eq. (4.3) has the form

$$\frac{d^2 r}{dx^2} + d_1(x)\frac{dr}{dx} + d_2(x) r = 0, \tag{4.12}$$

where

$$d_1(x) = \frac{B}{x(x+B)} + \frac{x}{x^2-1} + \frac{3}{2(x-x_0)}, \quad d_2(x) = \frac{x+B}{2(x^2-1)(x-x_0)}, \quad x_0 = -\frac{A + mR^2(B^2+1)}{2mR^2 B}.$$

After the change of variables (2.2) Eq. (4.12) becomes

$$\frac{d^2 y}{dx^2} = T_1(x) y, \tag{4.13}$$

where

$$\begin{aligned} T_1(x) = {} & \frac{\beta_1}{x-1} + \frac{\alpha_1}{(x-1)^2} + \frac{\beta_2}{x+1} + \frac{\alpha_2}{(x+1)^2} + \frac{\beta_3}{x-x_0} \\ & + \frac{\alpha_3}{(x-x_0)^2} + \frac{\beta_4}{x+B} + \frac{\alpha_4}{(x+B)^2} + \frac{\beta_5}{x} + \frac{\alpha_5}{x^2}, \end{aligned} \tag{4.14}$$

$$\alpha_1 = \alpha_2 = \alpha_3 = -\frac{3}{16}, \quad \alpha_4 = \frac{3}{4}, \quad \alpha_5 = -\frac{1}{4},$$

$$\beta_1 = \frac{4B^2 + (5x_0 - 3)B + x_0 - 3}{16(x_0 - 1)(B+1)}, \qquad \beta_2 = -\frac{4B^2 + (5x_0 + 3)B - x_0 - 3}{16(x_0 + 1)(B-1)},$$

$$\beta_3 = -\frac{2x_0 B^2 + (3 - 2x_0^2)B - x_0^3}{4x_0(x_0 + B)(x_0 - 1)(x_0 + 1)}, \qquad \beta_4 = \frac{7B^3 + 4x_0 B^2 - 5B - 2x_0}{4B(x_0 + B)(B+1)(B-1)},$$

$$\beta_5 = -\frac{3B + 2x_0}{4x_0 B}, \qquad x_0 = -\frac{A + mR^2(B^2+1)}{2mR^2 B}.$$

The Laurent expansion of $T_1(x)$ at $x = \infty$ in the considered case is

$$T_1(x)\big|_{x=\infty} \approx -\frac{3}{16x^2} + O\left(\frac{1}{x^3}\right).$$

Applying the Kovacic algorithm to the differential equation (4.13), we arrive at the following result.

Theorem 8. *Under the condition* (4.7)*, Eq.* (4.13) *has no Liouville solutions for all physically admissible values of parameters.*

Proof. First, we search for a solution of Eq. (4.13) of the form (2.4), i.e., a solution described in Case 1 of Theorem 1. Note that the function $T_1(x)$ has five finite second-order poles and a second-order pole at $x = \infty$. Therefore, all conditions of Theorem 4 are hold. Now we apply the Kovacic algorithm (see Sec. 2.3.1).

Step 1. Let us calculate the following values:

$$[\sqrt{T_1}]_1 = 0, \qquad \alpha_1^+ = \frac{3}{4}, \qquad \alpha_1^- = \frac{1}{4}, \qquad [\sqrt{T_1}]_{-B} = 0, \qquad \alpha_{-B}^+ = \frac{3}{2}, \qquad \alpha_{-B}^- = -\frac{1}{2},$$

$$[\sqrt{T_1}]_{-1} = 0, \qquad \alpha_{-1}^+ = \frac{3}{4}, \qquad \alpha_{-1}^- = \frac{1}{4}, \qquad [\sqrt{T_1}]_0 = 0, \qquad \alpha_0^+ = \frac{1}{2}, \qquad \alpha_0^- = \frac{1}{2},$$

$$[\sqrt{T_1}]_{x_0} = 0, \qquad \alpha_{x_0}^+ = \frac{3}{4}, \qquad \alpha_{x_0}^- = \frac{1}{4}, \qquad [\sqrt{T_1}]_\infty = 0, \qquad \alpha_\infty^+ = \frac{3}{4}, \qquad \alpha_\infty^- = \frac{1}{4}.$$

Step 2. Since the number ρ of finite poles of the function $T_1(x)$ is equal to 5, we have $2^{\rho+1} = 2^6 = 64$ tuples of signs

$$s = \big(s(\infty),\ s(1),\ s(-1),\ s(x_0),\ s(-B),\ s(0)\big).$$

For each tuple, we calculate the value d by the formula (2.19):

$$d = \alpha_\infty^{s(\infty)} - \alpha_1^{s(1)} - \alpha_{-1}^{s(-1)} - \alpha_{x_0}^{s(x_0)} - \alpha_{-B}^{s(-B)} - \alpha_0^{s(0)}.$$

According to the algorithm, d must be a nonnegative integer. Further, we analyze all possible tuples of signs s and corresponding values α. It is easy to verify that a unique tuple s such that d is nonnegative is

$$\alpha = \big(\alpha_\infty^+,\ \alpha_1^-,\ \alpha_{-1}^-,\ \alpha_{x_0}^-,\ \alpha_{-B}^-,\ \alpha_0^-\big) = \left(\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ -\frac{1}{2},\ \frac{1}{2}\right),$$

for which $d = 0$. The function $\theta = \theta(x)$ defined by (2.20) for the chosen tuple of values α has the form

$$\theta = \frac{1}{4(x-1)} + \frac{1}{4(x+1)} + \frac{1}{4(x-x_0)} - \frac{1}{2(x+B)} + \frac{1}{2x}.$$

Step 3. For the tuple of values α obtained on the previous step, we search for a polynomial of degree $d = 0$ ($P \equiv 1$) satisfying the differential equation (2.21). We substitute $P \equiv 1$ in (2.21) and get

$$\frac{B(Bx+1)}{2x(x+B)(x-x_0)(x^2-1)} = 0.$$

Obviously, this condition does not hold since $B > 0$. Therefore, Eq. (4.13) has no Liouville solutions of the form (2.4).

Now we search for a solution of the form (2.9) for the differential equation (4.13)., i.e. a solution described in Case 2 of Theorem 1. The necessary conditions of its existence are fulfilled (see Theorem 4). Now we apply the Kovacic algorithm (see Sec. 2.3.3).

Step 1. We define the following sets of integers:

$$E_1 = \{1,2,3\}, \qquad E_{-1} = \{1,2,3\}, \qquad E_{x_0} = \{1,2,3\}, \qquad E_\infty = \{1,2,3\},$$
$$E_{-B} = \{-2,2,6\}, \qquad E_0 = \{2\}.$$

Step 2. Consider all possible tuples of elements of E_∞, E_1, E_{-1}, E_{x_0}, E_{-B}, and E_0 with at least one odd element in each tuple:

$$s = \big(e_\infty,\ e_1,\ e_{-1},\ e_{x_0},\ e_{-B},\ e_0\big).$$

For each tuple s, we calculate d by (2.28):

$$d = \frac{1}{2}\big(e_\infty - e_1 - e_{-1} - e_{x_0} - e_{-B} - e_0\big).$$

According to the algorithm, d must be nonnegative integer. Consider all possible tuples of elements taken from E_∞, E_1, E_{-1}, E_{x_0}, E_{-B}, and E_0; the unique tuple with nonnegative d is

$$e = \big(e_\infty,\ e_1,\ e_{-1},\ e_{x_0},\ e_{-B},\ e_0\big) = \big(3,\ 1,\ 1,\ 1,\ -2,\ 2\big),$$

and $d = 0$.

Step 3. By the formula (2.29), we construct the function θ using the tuple e obtained on Step 2:

$$\theta = \frac{1}{2}\left(\frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-x_0} - \frac{2}{x+B} + \frac{2}{x}\right).$$

A polynomial of degree $d = 0$ ($P \equiv 1$) must satisfy Eq. (2.30). Substituting $P \equiv 1$ to (2.30), we obtain

$$-\frac{B(Bx^2 + (2-2B^2)x - B)}{x^2(x+B)^2(x^2-1)(x-x_0)} = 0.$$

Obviously, the last condition does not hold since $B > 0$. This means that Eq. (4.13) has no Liouville solutions of the form (2.9).

Now we search for a solution of the form (2.13) for Eq. (4.13), i.e., a solution described in Case 3 of Theorem 1. First, verify whether the necessary conditions of the existence of such a solution hold (see Theorem 4). The function $T_1(x)$ has no poles of order greater than 2. The order of $T_1(x)$ at ∞ is greater than 1. The partial fraction expansion of $T_1(x)$ has the form (4.14). Direct calculations show that all other conditions of Theorem 4 are satisfied:

$$\sqrt{1+4\alpha_i} = \frac{1}{2} \in \mathbb{Q}, \quad i = 1,2,3, \quad \sqrt{1+4\alpha_4} = 2 \in \mathbb{Q}, \quad \sqrt{1+4\alpha_5} = 0 \in \mathbb{Q},$$

$$\sum_{i=1}^{5} \beta_i = 0, \quad \sqrt{1+4\gamma} = \frac{1}{2} \in \mathbb{Q}, \quad \gamma = -\frac{3}{16}.$$

Now we apply the Kovacic algorithm (see Sec. 2.3.5).

Step 1. Let us define the following sets of integers:

$$E_1 = \{3,4,5,6,7,8,9\}, \qquad E_{-1} = \{3,4,5,6,7,8,9\},$$
$$E_{x_0} = \{3,4,5,6,7,8,9\}, \qquad E_{\infty} = \{3,4,5,6,7,8,9\},$$
$$E_{-B} = \{-6,-4,-2,0,2,4,6,8,10,12,14,16,18\}, \qquad E_0 = \{6\}.$$

Step 2. Now we consider all possible tuples

$$s = \bigl(e_\infty,\ e_1,\ e_{-1},\ e_{x_0},\ e_{-B},\ e_0\bigr)$$

of elements from E_∞, E_1, E_{-1}, E_{x_0}, E_{-B}, and E_0 and calculate d by (2.41):

$$d = e_\infty - e_1 - e_{-1} - e_{x_0} - e_{-B} - e_0.$$

According to the algorithm d must be nonnegative integer. Analyzing all possible tuples of elements taken from E_∞, E_1, E_{-1}, E_{x_0}, E_{-B}, and E_0, we conclude that the unique tuple with nonnegative d is

$$e = \bigl(e_\infty,\ e_1,\ e_{-1},\ e_{x_0},\ e_{-B},\ e_0\bigr) = \bigl(9,\ 3,\ 3,\ 3,\ -6,\ 6\bigr),$$

and $d = 0$.

Step 3. By (2.42) we construct the function θ using the tuple e obtained on the previous step. Then we get

$$\theta = \frac{3}{x-1} + \frac{3}{x+1} + \frac{3}{x-x_0} - \frac{6}{x+B} + \frac{6}{x}.$$

Using (2.43), we construct the polynomial

$$S = x(x-1)(x+1)(x-x_0)(x+B).$$

Further, the recursive formulas (2.44) are required:

$$P_{12} = -P, \quad P_{i-1} = -SP_i' + ((12-i)S' - S\theta)P_i - (12-i)(i+1)S^2T_1(x)P_{i+1}, \quad P_{-1} = 0,$$

where $P_{12} = -P \equiv -1$ is a polynomial of degree $d = 0$. The polynomial P_{-1} must be identically zero. Therefore, all its coefficients must be equal to zero. From this condition, using a computer algebra system, one can derive that either $B = 0$ or $x_0 = 0$. It was proved above that both these conditions have no physical sense. This means that Eq. (4.13) has no Liouville solutions of the form (2.13). Thus, we have proved that under the condition (4.7) Eq. (4.13) has no Liouville solutions. Theorem is proved. □

4.4. Special Case $A_3 = A_1 = m(a^2 - R^2)$. Assume that the moments of inertia of the torus satisfy the relation (4.9). Then the differential equation (4.3) has the following form:

$$\frac{d^2r}{dx^2} + d_1(x)\frac{dr}{dx} + d_2(x)r = 0, \tag{4.15}$$

where

$$d_1(x) = \frac{B}{x(x+B)} + \frac{x}{x^2-1} + \frac{3}{2(x+B)}, \quad d_2(x) = \frac{1}{2(x^2-1)}.$$

The change of variables (2.2) reduces Eq. (4.15) to the form

$$\frac{d^2y}{dx^2} = T_2(x)y, \tag{4.16}$$

where

$$T_2(x) = \frac{\beta_1}{x-1} + \frac{\alpha_1}{(x-1)^2} + \frac{\beta_2}{x+1} + \frac{\alpha_2}{(x+1)^2} + \frac{\beta_3}{x+B} + \frac{\alpha_3}{(x+B)^2} + \frac{\beta_4}{x} + \frac{\alpha_4}{x^2}, \tag{4.17}$$

$$\alpha_1 = \alpha_2 = \alpha_3 = -\frac{3}{16}, \quad \alpha_4 = -\frac{1}{4},$$

$$\beta_1 = \frac{B+3}{16(B+1)}, \quad \beta_2 = -\frac{B-3}{16(B-1)}, \quad \beta_3 = -\frac{2B^2-1}{4B(B^2-1)}, \quad \beta_4 = \frac{1}{4B}.$$

The Laurent expansion of $T_2(x)$ at $x = \infty$ is

$$T_2(x)\big|_{x=\infty} \approx -\frac{3}{16x^2} + O\left(\frac{1}{x^3}\right).$$

Direct application of the Kovacic algorithm to the differential equation (4.16) gives the following result.

Theorem 9. *Under the condition* (4.9)*, Eq.* (4.16) *has no Liouville solutions for all physically admissible values of parameters.*

Proof. First, we search for a solution of Eq. (4.16) of the form (2.4), i.e., a solution described in Case 1 of Theorem 1. Note that the function $T_2(x)$ has four finite second-order poles and a second-order pole at $x = \infty$. Therefore, the conditions of Theorem 4 are satisfied. Now we apply the Kovacic algorithm (see Sec. 2.3.1).

Step 1. Let us calculate the following values:

$$[\sqrt{T_2}]_1 = 0, \quad \alpha_1^+ = \frac{3}{4}, \quad \alpha_1^- = \frac{1}{4},$$

$$[\sqrt{T_2}]_{-1} = 0, \quad \alpha_{-1}^+ = \frac{3}{4}, \quad \alpha_{-1}^- = \frac{1}{4},$$

$$[\sqrt{T_2}]_{-B} = 0, \quad \alpha_{-B}^+ = \frac{3}{4}, \quad \alpha_{-B}^- = \frac{1}{4},$$

$$[\sqrt{T_2}]_0 = 0, \quad \alpha_0^+ = \frac{1}{2}, \quad \alpha_0^- = \frac{1}{2},$$

$$[\sqrt{T_2}]_\infty = 0, \quad \alpha_\infty^+ = \frac{3}{4}, \quad \alpha_\infty^- = \frac{1}{4}.$$

Step 2. Since the number ρ of finite poles of the function $T_2(x)$ is equal to 4, we have $2^{\rho+1} = 2^5 = 32$ tuples of signs

$$s = \big(s(\infty),\ s(1),\ s(-1),\ s(-B),\ s(0)\big).$$

For each tuple s, we calculate d by (2.19):

$$d = \alpha_{\infty}^{s(\infty)} - \alpha_{1}^{s(1)} - \alpha_{-1}^{s(-1)} - \alpha_{-B}^{s(-B)} - \alpha_{0}^{s(0)}.$$

According to the algorithm, d must be nonnegative integer. However, a direct calculation shows that $d = -1/2$ is the largest possible value for d. Hence the differential equation (4.16) has no Liouville solutions of the form (2.4).

Now we search for a solution of the form (2.9) for the differential equation (4.16), i.e., a solution described in Case 2 of Theorem 1. The necessary conditions for existence of a such solution hold (see Theorem 4). Now we apply the Kovacic algorithm (see Sec. 2.3.3).

Step 1. Let us define the following sets of integers:

$$E_1 = \{1, 2, 3\}, \quad E_{-1} = \{1, 2, 3\}, \quad E_{-B} = \{1, 2, 3\}, \quad E_{\infty} = \{1, 2, 3\}, \quad E_0 = \{2\}.$$

Step 2. Now we consider all tuples

$$s = \bigl(e_{\infty},\ e_1,\ e_{-1},\ e_{-B},\ e_0\bigr)$$

of elements from E_{∞}, E_1, E_{-1}, E_{-B}, and E_0; in each tuple at least one element is odd. For each tuple s, we calculate d by (2.28):

$$d = \frac{1}{2}\bigl(e_{\infty} - e_1 - e_{-1} - e_{-B} - e_0\bigr).$$

According to the algorithm, d must be nonnegative integer. A direct calculation shows that $d = -1$ is the largest possible value for d. Hence Eq. (4.16) has no Liouville solutions of the form (2.9).

Now we search for a solution of the form (2.13) for Eq. (4.16), i.e., a solution described in Case 3 of Theorem 1. First, we verify whether the necessary conditions for its existence hold (see Theorem 4). The function $T_2(x)$ has no poles of order greater than 2. The order of the pole of $T_2(x)$ at $x = \infty$ is greater than 1. The partial fraction expansion of $T_2(x)$ is (4.17). It can be easily proved that the remaining conditions of Theorem 4 are also fulfilled:

$$\sqrt{1 + 4\alpha_i} = \frac{1}{2} \in \mathbb{Q} \quad (i = 1, 2, 3), \qquad \sqrt{1 + 4\alpha_4} = 0 \in \mathbb{Q},$$

$$\sum_{i=1}^{4} \beta_i = 0, \qquad \sqrt{1 + 4\gamma} = \frac{1}{2} \in \mathbb{Q}, \qquad \gamma = -\frac{3}{16}.$$

Now we apply the Kovacic algorithm step (see Sec. 2.3.5).

Step 1. Let us define the following sets of integers:

$$\begin{aligned} &E_1 = \{3, 4, 5, 6, 7, 8, 9\}, && E_{-1} = \{3, 4, 5, 6, 7, 8, 9\}, \\ &E_{-B} = \{3, 4, 5, 6, 7, 8, 9\}, && E_{\infty} = \{3, 4, 5, 6, 7, 8, 9\}, \\ &E_0 = \{6\}. \end{aligned}$$

Step 2. Consider all possible tuples

$$s = \bigl(e_{\infty},\ e_1,\ e_{-1},\ e_{-B},\ e_0\bigr)$$

of elements from E_{∞}, E_1, E_{-1}, E_{-B}, and E_0 and calculate d by (2.41):

$$d = e_{\infty} - e_1 - e_{-1} - e_{-B} - e_0.$$

According to the algorithm, d must be nonnegative integer. A direct calculation shows that $d = -6$ is the largest possible value for d. Hence Eq. (4.16) has no Liouville solutions of the form (2.13).

Thus, we can conclude that under the condition (4.9), Eq. (4.16) has no Liouville solutions. The theorem is proved. $\square$

4.5. Special Case $A_1 = m(a^2 - R^2) \neq A_3$. Assume that the moments of inertia of the torus satisfy the relation (4.10). Then the differential equation (4.3) has the form

$$\frac{d^2r}{dx^2} + d_1(x)\frac{dr}{dx} + d_2(x)r = 0, \tag{4.18}$$

where

$$d_1(x) = \frac{B}{x(x+B)} + \frac{x}{x^2-1} + \frac{3(2x - x_0 + B)}{2(x+B)(x-x_0)}, \quad d_2(x) = \frac{4x^2 - (x_0+B)x - 2}{2x(x^2-1)(x-x_0)},$$
$$x_0 = \frac{(A_3 + mR^2B^2 - mR^2)B}{A_3 + mR^2 - mR^2B^2}.$$

The change of variables (2.2) reduces the differential equation (4.18) to the form

$$\frac{d^2y}{dx^2} = T_3(x)y, \tag{4.19}$$

where

$$T_3(x) = \frac{\beta_1}{x-1} + \frac{\alpha_1}{(x-1)^2} + \frac{\beta_2}{x+1} + \frac{\alpha_2}{(x+1)^2} + \frac{\beta_3}{x+B} + \frac{\alpha_3}{(x+B)^2} + \frac{\beta_4}{x-x_0} + \frac{\alpha_4}{(x-x_0)^2} + \frac{\beta_5}{x} + \frac{\alpha_5}{x^2} \tag{4.20}$$

and

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\frac{3}{16}, \quad \alpha_5 = -\frac{1}{4},$$

$$\beta_1 = -\frac{4B^2 - (x_0-7)B - 3x_0 + 5}{16(B+1)(x_0-1)}, \qquad \beta_2 = \frac{4B^2 - (x_0+7)B + 3x_0 + 5}{16(B-1)(x_0+1)},$$
$$\beta_3 = -\frac{7B^3 + 4x_0B^2 - 5B - 2x_0}{8B(B^2-1)(x_0+B)}, \qquad \beta_4 = \frac{4x_0B^2 + (4x_0^2+2)B + 3x_0^3 - x_0}{8x_0(x_0^2-1)(x_0+B)},$$
$$\beta_5 = \frac{x_0+B}{4Bx_0}.$$

As was noted above, in the case considered, all finite poles of $T_3(x)$ are distinct. The Laurent expansion of $T_3(x)$ at $x = \infty$ has the form

$$T_3(x)\big|_{x=\infty} \approx \frac{5B^2 + 2x_0B - 3x_0^2 - 8}{16x^4} + O\left(\frac{1}{x^5}\right).$$

Direct application of the Kovacic algorithm to the differential equation (4.19) yields the following result.

Theorem 10. *Under the condition* (4.10), *Eq.* (4.19) *has no Liouville solutions for all physically admissible values of parameters.*

Proof. First, we search for a solution of Eq. (4.19) of the form (2.4), i.e., a solution described in Case 1 of Theorem 1. Note that the function $T_3(x)$ has five finite second-order poles and a fourth-order pole at $x = \infty$ (i.e., at infinity the order is greater than 2). Therefore, the conditions of Theorem 4 are satisfied. Now we apply the Kovacic algorithm (see Sec. 2.3.1).

Step 1. Let us calculate the following values:

$$[\sqrt{T_3}]_1 = 0, \qquad \alpha_1^+ = \frac{3}{4}, \qquad \alpha_1^- = \frac{1}{4}, \qquad [\sqrt{T_3}]_{x_0} = 0, \qquad \alpha_{x_0}^+ = \frac{3}{4}, \qquad \alpha_{x_0}^- = \frac{1}{4},$$
$$[\sqrt{T_3}]_{-1} = 0, \qquad \alpha_{-1}^+ = \frac{3}{4}, \qquad \alpha_{-1}^- = \frac{1}{4}, \qquad [\sqrt{T_3}]_0 = 0, \qquad \alpha_0^+ = \frac{1}{2}, \qquad \alpha_0^- = \frac{1}{2},$$
$$[\sqrt{T_3}]_{-B} = 0, \qquad \alpha_{-B}^+ = \frac{3}{4}, \qquad \alpha_{-B}^- = \frac{1}{4}, \qquad [\sqrt{T_3}]_\infty = 0, \qquad \alpha_\infty^+ = 0, \qquad \alpha_\infty^- = 1.$$

Step 2. Since the number ρ of finite poles of the function $T_3(x)$ is 5, we have $2^{\rho+1} = 2^6 = 64$ tuples of signs

$$s = \big(s(\infty),\ s(1),\ s(-1),\ s(x_0),\ s(-B),\ s(0)\big).$$

For each tuple s, we calculate d by (2.19):

$$d = \alpha_\infty^{s(\infty)} - \alpha_1^{s(1)} - \alpha_{-1}^{s(-1)} - \alpha_{x_0}^{s(x_0)} - \alpha_{-B}^{s(-B)} - \alpha_0^{s(0)}.$$

According to the algorithm, d must be nonnegative integer. A direct calculation shows that $d = -1/2$ is the largest possible value for d. Hence Eq. (4.19) has no Liouville solutions of the form (2.4).

Now we search for a solution of the form (2.9) for the differential equation (4.19), i.e., a solution described in Case 2 of Theorem 1. The necessary conditions for the existence of a such solution are fulfilled (see Theorem 4). Now we apply the Kovacic algorithm (see Sec. 2.3.3).

Step 1. Let us define the following sets of integers:

$$E_1 = \{1, 2, 3\}, \qquad E_{-1} = \{1, 2, 3\}, \qquad E_{-B} = \{1, 2, 3\}, \qquad E_{x_0} = \{1, 2, 3\},$$
$$E_0 = \{2\}, \qquad E_\infty = \{0, 2, 4\}$$

Step 2. Now we consider all possible tuples

$$s = \big(e_\infty,\ e_1,\ e_{-1},\ e_{x_0},\ e_{-B},\ e_0\big)$$

of elements of E_∞, E_1, E_{-1}, E_{x_0}, E_{-B}, and E_0; in each tuple at least one element is odd. For each tuple s, we calculate d by (2.28):

$$d = \frac{1}{2}\big(e_\infty - e_1 - e_{-1} - e_{x_0} - e_{-B} - e_0\big).$$

According to the algorithm, d must be a nonnegative integer. However, direct calculations show that $d = -1$ is the largest possible value for d. Hence Eq. (4.19) has no Liouville solutions of the form (2.9).

Now we search for a solution of the form (2.13) for the differential equation (4.19), i.e., a solution described in Case 3 of Theorem 1. First, we verify the fulfillment of the necessary conditions (see Theorem 4). The function $T_3(x)$ has no poles of order greater than 2. The order of the pole of $T_3(x)$ at $x = \infty$ is greater than 1. The partial fraction expansion of $T_3(x)$ is (4.20). We can easily show that the remaining conditions of Theorem 4 hold:

$$\sqrt{1 + 4\alpha_i} = \frac{1}{2} \in \mathbb{Q} \quad (i = 1, \ldots, 4), \qquad \sqrt{1 + 4\alpha_5} = 0 \in \mathbb{Q},$$
$$\sum_{i=1}^{5} \beta_i = 0, \qquad \sqrt{1 + 4\gamma} = 1 \in \mathbb{Q}, \qquad \gamma = 0.$$

Now we apply the Kovacic algorithm (see Sec. 2.3.5).

Step 1. Let us define the following sets of integers:

$$E_1 = \{3,4,5,6,7,8,9\}, \qquad E_{-1} = \{3,4,5,6,7,8,9\},$$
$$E_{-B} = \{3,4,5,6,7,8,9\}, \qquad E_{x_0} = \{3,4,5,6,7,8,9\},$$
$$E_0 = \{6\}, \qquad E_\infty = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}.$$

Step 2. Now we consider all possible tuples

$$s = \big(e_\infty,\ e_1,\ e_{-1},\ e_{x_0},\ e_{-B},\ e_0\big)$$

of elements from E_∞, E_1, E_{-1}, E_{x_0}, E_{-B}, and E_0 and calculate d by (2.41):

$$d = e_\infty - e_1 - e_{-1} - e_{x_0} - e_{-B} - e_0.$$

According to the algorithm, d must be nonnegative integer. However, a direct calculation shows that $d = -6$ is the largest possible value for d. Hence Eq. (4.19) has no Liouville solutions of the form (2.13).

Thus, we conclude that under the condition (4.10), Eq. (4.19) has no Liouville solutions. The theorem is proved. □

4.6. Special case $A_3(A_1 + mR^2) = A_1(A_1 + mR^2 - ma^2)$. Assume that the moments of inertia of the torus satisfy the relation (4.11). Then Eq. (4.3) has the form

$$\frac{d^2r}{dx^2} + d_1(x)\frac{dr}{dx} + d_2(x)r = 0, \tag{4.21}$$

where

$$d_1(x) = \frac{B}{x(x+B)} + \frac{x}{x^2-1} + \frac{3}{x-x_0}, \quad d_2(x) = \frac{(x+B)(2x^2-(x_0+B)x-1)}{x(x^2-1)(x-x_0)^2},$$
$$x_0 = -\frac{A_1+mR^2}{mR^2B}.$$

The change of variables (2.2) reduces the differential equation (4.21) to the form

$$\frac{d^2y}{dx^2} = T_4(x)y, \tag{4.22}$$

where

$$T_4(x) = \frac{\beta_1}{x-1} + \frac{\alpha_1}{(x-1)^2} + \frac{\beta_2}{x+1} + \frac{\alpha_2}{(x+1)^2} + \frac{\beta_3}{x-x_0} + \frac{\alpha_3}{(x-x_0)^2} + \frac{\beta_4}{x+B} + \frac{\alpha_4}{(x+B)^2} + \frac{\beta_5}{x} + \frac{\alpha_5}{x^2}, \tag{4.23}$$

$$\alpha_1 = \alpha_2 = -\frac{3}{16}, \quad \alpha_3 = \frac{4x_0B^2+4B-x_0^3+x_0}{4x_0(x_0^2-1)}, \quad \alpha_4 = \frac{3}{4}, \quad \alpha_5 = -\frac{1}{4}.$$

$$\beta_1 = \frac{8B^3+8(x_0+1)B^2+(5x_0^2-6x_0+9)B+x_0^2-6x_0+5}{16(x_0-1)^2(B+1)},$$

$$\beta_2 = -\frac{8B^3+8(x_0-1)B^2+(5x_0^2+6x_0+9)B-x_0^2-6x_0-5}{16(x_0+1)^2(B-1)},$$

$$\beta_3 = -\frac{4x_0^3B^3+(6x_0^4+4x_0^2-2)B^2-(2x_0^5-11x_0^3+5x_0)B-x_0^6+x_0^4}{2x_0^2(x_0^2-1)^2(x_0+B)},$$

$$\beta_4 = \frac{5B^3+2x_0B^2-4B-x_0}{2B(B^2-1)(x_0+B)}, \qquad \beta_5 = -\frac{2B^2+3x_0B+x_0^2}{2Bx_0^2}.$$

The Laurent expansion of $T_4(x)$ at $x = \infty$ has the form

$$T_4(x)\big|_{x=\infty} \approx \frac{3B^2 + 2x_0 B - x_0^2 - 2}{4x^4} + O\left(\frac{1}{x^5}\right).$$

The special case considered has the following peculiarity: one of the coefficients α_i of the partial fraction expansion of the function $T_4(x)$ depends on parameters. The coefficient α_3 has no definite numerical value but it is determined by the expression

$$\alpha_3 = \frac{4x_0 B^2 + 4B - x_0^3 + x_0}{4x_0(x_0^2 - 1)}.$$

As a result, the constant d can be arbitrarily large. (Recall that d is the degree of a polynomial P calculated in each case.) We restrict ourselves to considering only the case where $d = 0$. Direct application of the Kovacic algorithm to the differential equation (4.22) yields the following result.

Theorem 11. *Assume that $d = 0$ and the condition* (4.11) *is fulfilled. Then the differential equation* (4.22) *has no Liouville solutions for all physically admissible values of parameters.*

Proof. First, we search for a solution of Eq. (4.22) of the form (2.4), i.e., a solution described in Case 1 of Theorem 1. Note that the function $T_4(x)$ has five finite second-order poles and a fourth-order pole at $x = \infty$. Therefore, the conditions of Theorem 4 are satisfied. Now we apply the Kovacic algorithm (see Sec. 2.3.1).

Step 1. Let $b_0 = 1 + 4\alpha_3$. We calculate the following values:

$$\begin{array}{lll}
[\sqrt{T_4}]_1 = 0, & \alpha_1^+ = \dfrac{3}{4}, & \alpha_1^- = \dfrac{1}{4}, \\
[\sqrt{T_4}]_{-1} = 0, & \alpha_{-1}^+ = \dfrac{3}{4}, & \alpha_{-1}^- = \dfrac{1}{4}, \\
[\sqrt{T_4}]_{-B} = 0, & \alpha_{-B}^+ = \dfrac{3}{2}, & \alpha_{-B}^- = -\dfrac{1}{2}, \\
[\sqrt{T_4}]_0 = 0, & \alpha_0^+ = \dfrac{1}{2}, & \alpha_0^- = \dfrac{1}{2}, \\
[\sqrt{T_4}]_{x_0} = 0, & \alpha_{x_0}^+ = \dfrac{1}{2} + \dfrac{1}{2}\sqrt{b_0}, & \alpha_{x_0}^- = \dfrac{1}{2} - \dfrac{1}{2}\sqrt{b_0}, \\
[\sqrt{T_4}]_\infty = 0, & \alpha_\infty^+ = 0, & \alpha_\infty^- = 1.
\end{array}$$

Step 2. Since the number ρ of finite poles of the function $T_4(x)$ is equal to 5, we have $2^{\rho+1} = 2^6 = 64$ tuples of signs

$$s = \big(s(\infty),\ s(1),\ s(-1),\ s(-B),\ s(0),\ s(x_0)\big).$$

Choose signs in the tuples s in a such way that the value

$$d = \alpha_\infty^{s(\infty)} - \alpha_1^{s(1)} - \alpha_{-1}^{s(-1)} - \alpha_{-B}^{s(-B)} - \alpha_0^{s(0)} - \alpha_{x_0}^{s(x_0)},$$

calculated by (2.19), is equal to zero for some b_0. All such tuples of signs and corresponding values b_0 are listed below:

$$
\begin{array}{llll}
s_1 = (-,+,+,+,+,-), & b_0 = 36; & s_{10} = (+,+,+,+,+,-), & b_0 = 64; \\
s_2 = (-,+,+,-,+,-), & b_0 = 4; & s_{11} = (+,+,+,-,+,-), & b_0 = 16; \\
s_3 = (-,+,-,+,+,-), & b_0 = 25; & s_{12} = (+,+,-,+,+,-), & b_0 = 49; \\
s_4 = (-,-,+,+,+,-), & b_0 = 25; & s_{13} = (+,-,+,+,+,-), & b_0 = 49; \\
s_5 = (-,-,-,+,+,-), & b_0 = 16; & s_{14} = (+,-,-,+,+,-), & b_0 = 36; \\
s_6 = (-,-,+,-,+,-), & b_0 = 1; & s_{15} = (+,-,+,-,+,-), & b_0 = 9; \\
s_7 = (-,+,-,-,+,-), & b_0 = 1; & s_{16} = (+,+,-,-,+,-), & b_0 = 9; \\
s_8 = (-,-,-,-,+,-), & b_0 = 0; & s_{17} = (+,-,-,-,+,-), & b_0 = 4. \\
s_9 = (-,-,-,-,+,+), & b_0 = 0; & &
\end{array}
$$

Let us consider in more detail the case where the tuple s_1 is chosen; the other cases can be considered similarly. Let us solve the equation

$$b_0 = \frac{4B(Bx_0+1)}{x_0(x_0^2-1)} = 36$$

with respect to B. Since $B > 1$ and $x_0 < -1$, we have

$$B = \frac{-1-\sqrt{36x_0^4-36x_0^2+1}}{2x_0}. \tag{4.24}$$

Further, using the formula (2.20), we construct the function $\theta(x)$ using the values $\alpha_c^{\pm}$ corresponding to the signs chosen for the tuple s_1. Then the function θ has the form

$$\theta = \frac{3}{4(x-1)} + \frac{3}{4(x+1)} + \frac{3}{2(x+B)} + \frac{1}{2x} + \frac{1-\sqrt{b_0}}{2(x-x_0)}.$$

Step 3. A polynomial of degree $d = 0$ ($P \equiv 1$) must satisfy Eq. (2.21) identically. Eliminating B from Eq. (2.21) using the formula (4.24) we obtain

$$\frac{(18x_0^5 - 9x_0^3 - 7x_0^3x_*)x + 22x_0^2(x_0^2-1) + x_*(1-4x_0^2) + 1}{x(x^2-1)(x-x_0)(2x_0x - x_* - 1)x_0^2} = 0, \tag{4.25}$$

where

$$x_* = \sqrt{36x_0^4 - 36x_0^2 + 1}.$$

Since (4.25) is an identity, for some values of x_0 ($x_0 < -1$) the following conditions hold:

$$18x_0^5 - 9x_0^3 - 7x_0^3\sqrt{36x_0^4-36x_0^2+1} = 0, \quad 22x_0^2(x_0^2-1) + (1-4x_0^2)\sqrt{36x_0^4-36x_0^2+1} + 1 = 0.$$

It is easy to verify that this system of equations has no solutions such that $x_0 < -1$. Consequently, for the tuple of signs s_1, Eq. (4.22) has no Liouville solutions of the form (2.4). Similarly, we can consider all remaining tuples of signs and ascertain that Eq. (4.22) does not possess any Liouville solution of the form (2.4).

Now we search for a solution of the form (2.9) for the differential equation (4.22), i.e., a solution described in Case 2 of Theorem 1. The necessary conditions for the existence of such a solution are fulfilled (see Theorem 4). Now we apply the Kovacic algorithm (see Sec. 2.3.3).

Step 1. Let us define the following sets of integers:

$$E_1 = \{1, 2, 3\}, \quad E_{-1} = \{1, 2, 3\}, \quad E_{-B} = \{-2, 2, 6\}, \quad E_0 = \{2\},$$
$$E_{x_0} = \left\{(2 + k\sqrt{b_0}) \cap \mathbb{Z},\ k = 0, \pm 2\right\}, \quad E_\infty = \{0, 2, 4\}.$$

Step 2. Consider all possible tuples

$$s = \bigl(e_\infty,\ e_1,\ e_{-1},\ e_{-B},\ e_0,\ e_{x_0}\bigr)$$

of elements from E_∞, E_1, E_{-1}, E_{-B}, E_0, and E_{x_0}; in each tuple at least one element is odd. For each set s, we calculate d by (2.28):

$$d = \frac{1}{2}\bigl(e_\infty - e_1 - e_{-1} - e_{-B} - e_0 - e_{x_0}\bigr).$$

As before, we assume that $d = 0$. Execution of this step of the algorithm involves a large number of possibilities. Thus, we present here a detailed investigation for only one case. All other cases can be studied in the same way. Choose the tuple

$$s_1 = \bigl(e_\infty,\ e_1,\ e_{-1},\ e_{-B},\ e_0,\ e_{x_0}\bigr) = \bigl(0,\ 1,\ 1,\ 2,\ 2,\ -6\bigr).$$

In this case, we have $b_0 = 16$. Therefore,

$$B = \frac{-1 - \sqrt{16x_0^4 - 16x_0^2 + 1}}{2x_0}.$$

Step 3. By the formula (2.29), we form the function θ using elements of the tuple s_1. Hence θ has the form

$$\theta = \frac{1}{2(x-1)} + \frac{1}{2(x+1)} + \frac{1}{x+B} + \frac{1}{x} - \frac{3}{x - x_0}.$$

A polynomial of degree $d = 0$ ($P \equiv 1$) must satisfy equation (2.30). We substitute $P \equiv 1$ to (2.30) and obtain

$$\frac{k_5x^5 + k_4x^4 + k_3x^3 + k_2x^2 + k_1x + k_0}{x_0^2x^2(x^2-1)(x-x_0)^3(2x_0x - x_* - 1)^2} = 0, \tag{4.26}$$

where

$$k_0 = 96x_0^8 - (4x_* + 148)x_0^6 + (12x_* + 60)x_0^4 - (x_*^3 + 3x_* + 4)x_0^2,$$
$$k_1 = 256x_0^9 + 256x_0^7 - (60x_* + 780)x_0^5 - (2x_*^3 - 102x_* - 308)x_0^3 - (3x_*^3 + 17x_* + 20)x_0,$$
$$k_2 = 480x_0^8 + (24x_* - 1008)x_0^6 - (180x_* - 828)x_0^4 - (9x_*^3 - 45x_* + 348)x_0^2 + 12x_*^3 + 12x_* + 24,$$
$$k_3 = -512x_0^7 + (120x_* + 512)x_0^5 - (140x_* - 100)x_0^3 - (5x_*^3 + 15x_* + 20)x_0,$$
$$k_4 = -152x_0^6 + (100x_* + 132)x_0^4, \quad k_5 = -72x_0^5 + (12x_* + 12)x_0^3, \quad x_* = \sqrt{16x_0^4 - 16x_0^2 + 1}.$$

For (4.26) to be an identity, the following conditions are necessary: $k_i = 0$, $i = 0, 1, \ldots, 5$. However, we can easily verify that this system of equations with respect to the unknown constant x_0 is inconsistent. Thus, we have proved that for the tuple s_1 of elements from E_∞, E_1, E_{-1}, E_{-B}, E_0, and E_{x_0}, the differential equation (4.22) has no Liouville solution of the form (2.9). Note that a similar investigation was conducted for all other tuples s with $d = 0$. As a result, we prove the nonexistence of solutions of the form (2.9) for the differential equation (4.22) for any tuple s.

Finally, we search for a solution of the form (2.13) for the differential equation (4.22), i.e., a solution described in Case 3 of Theorem 1. The necessary conditions (see Theorem 4) are fulfilled. The function $T_4(x)$ has no poles of order greater than 2. The order of pole of $T_4(x)$ at $x = \infty$ is greater than 1. The partial fraction expansion of $T_4(x)$ is (4.23). It can be easily shown that the remaining conditions of Theorem 4 hold:

$$\sqrt{1+4\alpha_i} = \frac{1}{2} \in \mathbb{Q} \quad (i=1,2), \qquad \sqrt{1+4\alpha_4} = 2 \in \mathbb{Q}, \qquad \sqrt{1+4\alpha_5} = 0 \in \mathbb{Q},$$
$$\sum_{i=1}^{5} \beta_i = 0, \qquad \sqrt{1+4\gamma} = 1 \in \mathbb{Q}, \qquad \gamma = 0.$$

Let us assume that the condition

$$\sqrt{1+4\alpha_3} = \sqrt{\frac{4B(Bx_0+1)}{x_0(x_0^2-1)}} \in \mathbb{Q}$$

is valid. Otherwise Eq. (4.22) obviously has no Liouville solutions of the form (2.13). Now we apply the Kovacic algorithm (see Sec. 2.3.5).

Step 1. Let us define the sets

$$E_1 = \{3,4,5,6,7,8,9\}, \qquad E_{-1} = \{3,4,5,6,7,8,9\},$$
$$E_{-B} = \big\{-6,\ -4,\ -2,\ 0,\ 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18\big\}, \qquad E_0 = \{6\},$$
$$E_{x_0} = \Big\{(6+k\sqrt{b_0}) \cap \mathbb{Z},\ k=0,\ \pm 1,\ \ldots,\ \pm 6)\Big\},$$
$$E_\infty = \big\{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12\big\}.$$

Step 2. Consider all possible tuples

$$s = \big(e_\infty,\ e_1,\ e_{-1},\ e_{-B},\ e_0,\ e_{x_0}\big)$$

of elements from E_∞, E_1, E_{-1}, E_{-B}, E_0, and E_{x_0} and calculate d by (2.41):

$$d = e_\infty - e_1 - e_{-1} - e_{-B} - e_0 - e_{x_0}.$$

Step 3. We assume $d=0$ as before. Among all tuples s, we choose tuples with $d=0$. Considering numbers of elements of sets E_∞, E_1, E_{-1}, E_{-B}, E_0, and E_{x_0} obtained on the first step, we can estimate that even if we fix one element of E_{x_0}, we must investigate $7\cdot 7\cdot 13\cdot 13 = 8281$ tuples s, for each of which $d=0$. Therefore, we shall illustrate the investigation on a typical example instead of listing all possible cases. Choose a tuple s_1 with $d=0$:

$$s_1 = \big(e_\infty,\ e_1,\ e_{-1},\ e_{-B},\ e_0,\ e_{x_0}\big) = \big(12,\ 3,\ 3,\ 0,\ 6,\ 0\big).$$

By the formula (2.42), we construct the function θ using the tuple s_1. The function θ has the form

$$\theta = \frac{3}{x-1} + \frac{3}{x+1} + \frac{6}{x}.$$

According to (2.43), we construct the polynomial

$$S = x(x-1)(x+1)(x-x_0)(x+B),$$

where B is expressed in terms of x_0:

$$B = \frac{-1-\sqrt{36x_0^4 - 36x_0^2 + 1}}{2x_0}.$$

Further, the recursive formulas (2.44) are required:

$$P_{12} = -P, \quad P_{i-1} = -SP_i' + ((12-i)S' - S\theta)P_i - (12-i)(i+1)S^2 T_4(x) P_{i+1}, \quad P_{-1} = 0,$$

where $P_{12} = -P \equiv -1$ is a polynomial of degree $d=0$. The polynomial P_{-1} must be identically zero. Therefore, all its coefficients must be equal to zero. These coefficients include one unknown

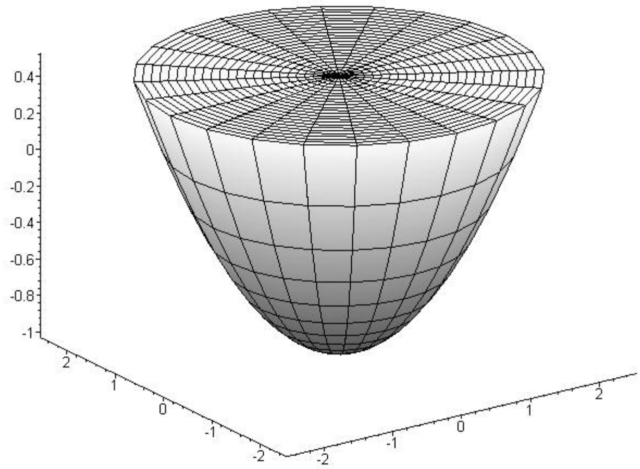


Fig. 5. Dynamically symmetric paraboloid on the horizontal plane.

variable x_0. The corresponding system of equations is inconsistent. Similarly, all other tuples s may be considered. This means that Eq. (4.22) has no Liouville solutions of the form (2.13) in the case $d = 0$. The theorem is proved. $\square$

In summary, the above investigation shows the absence of Liouville solutions in the problem of rolling of a dynamically symmetric torus on a perfectly rough plane. The conclusion is valid in the general case as well as in the special cases where the parameters satisfy additional conditions.

5. Motion of a Rotationally Symmetric Paraboloid

5.1. Formulation of the problem. Equations of motion. In this section, we consider the problem of the motion of a rotationally symmetric paraboloid with focal length 2λ on a perfectly rough horizontal plane (Fig. 5). We assume that the center of mass G of the paraboloid coincides with the focus of the generating parabola. Then the distance between the center of mass and the horizontal supporting plane is

$$f(\theta) = \frac{\lambda}{\cos\theta}. \tag{5.1}$$

Using (3.11), we calculate the coordinates ξ and ζ of the point of contact of the paraboloid with the horizontal plane:

$$\xi = -\frac{2\lambda\sin\theta}{\cos\theta}, \quad \zeta = \frac{\lambda\sin^2\theta}{\cos^2\theta} - \lambda, \quad \zeta = \frac{\xi^2}{4\lambda} - \lambda. \tag{5.2}$$

The system (3.29) has the form

$$\begin{cases} \dfrac{dp}{d\theta} = -\left(\dfrac{\cos\theta}{\sin\theta} + \dfrac{2A_3m\lambda^2\sin^3\theta(1-2\cos^2\theta)}{\Delta\cos\theta}\right)p \\ \qquad\qquad + \dfrac{A_3((A_3-4m\lambda^2)\cos^4\theta + 8m\lambda^2\cos^2\theta - 2m\lambda^2)}{\Delta}r, \\ \dfrac{dr}{d\theta} = -\dfrac{4A_1m\lambda^2\sin^4\theta}{\Delta}p - \dfrac{2m\lambda^2\sin\theta\cos\theta(2A_1+A_3-2A_3\cos^2\theta)}{\Delta}r, \\ \Delta = (A_1A_3 + 4m\lambda^2(A_3-A_1))\cos^4\theta - 4m\lambda^2(A_3-A_1)\cos^2\theta + A_3m\lambda^2, \end{cases} \tag{5.3}$$

and Eq. (3.30) can be written as follows:

$$\frac{d^2r}{d\theta^2}+b_1\frac{dr}{d\theta}+b_2r=0, \tag{5.4}$$

where

$$b_1=\frac{\cos^2\theta-4}{\sin\theta\cos\theta}+\frac{6(A_3-2(A_3-A_1)\cos^2\theta)m\lambda^2\sin\theta}{\Delta\cos\theta},\quad b_2=\frac{2m\lambda^2(A_3-2A_1)(1+\cos^2\theta)}{\Delta}.$$

Note that under the condition

$$A_3=2A_1, \tag{5.5}$$

Eq. (5.4) has the partial solution $r=r_0=\text{const}$. This fact was firstly obtained by Kh. M. Mushtari (see [35]). In Eq. (5.4), we perform the change of the independent variable by the formula $\cos^2\theta=x$ and denote $B=m\lambda^2$. Hence we obtain

$$\frac{d^2r}{dx^2}+d_1(x)\frac{dr}{dx}+d_2(x)r=0, \tag{5.6}$$

where

$$d_1(x)=\frac{5-3x}{2x(1-x)}-\frac{3(A_3-2(A_3-A_1)x)B}{x\Delta},\quad d_2(x)=\frac{(A_3-2A_1)B(x+1)}{2x(1-x)\Delta},$$
$$\Delta=(A_1A_3+4(A_3-A_1)B)x^2-4(A_3-A_1)Bx+A_3B.$$

In the general case, the polynomial Δ in the expressions d_1 and d_2 has two roots x_1 and x_2. In the explicit form they can be written as follows:

$$\begin{aligned}x_1&=\frac{2B(A_3-A_1)-\sqrt{4A_1B^2(A_1-A_3)-A_1A_3^2B}}{A_1A_3+4B(A_3-A_1)},\\ x_2&=\frac{2B(A_3-A_1)+\sqrt{4A_1B^2(A_1-A_3)-A_1A_3^2B}}{A_1A_3+4B(A_3-A_1)}.\end{aligned} \tag{5.7}$$

After the change of variables (2.2) in (5.6) we obtain

$$\frac{d^2y}{dx^2}=\Pi(x)y, \tag{5.8}$$

where

$$\Pi(x)=\frac{\beta_1}{x-1}+\frac{\alpha_1}{(x-1)^2}+\frac{\beta_0}{x}+\frac{\alpha_0}{x^2}+\frac{\beta_2}{x-x_1}+\frac{\alpha_2}{(x-x_1)^2}+\frac{\beta_3}{x-x_2}+\frac{\alpha_3}{(x-x_2)^2},$$
$$\alpha_0=\frac{5}{16},\quad \alpha_1=\frac{3}{4},\quad \alpha_2=\alpha_3=-\frac{3}{16},$$
$$\beta_0=\frac{x_1+x_2+2x_1x_2}{8x_1x_2},\qquad \beta_2=-\frac{4x_1+x_2-7x_1x_2-2x_1^2+4x_1^2x_2}{8x_1(x_1-x_2)(x_1-1)},$$
$$\beta_1=\frac{4x_1+4x_2-3x_1x_2-5}{4(x_1-1)(x_2-1)},\qquad \beta_3=\frac{x_1+4x_2-7x_1x_2-2x_2^2+4x_1x_2^2}{8x_2(x_1-x_2)(x_2-1)}.$$

We apply the Kovacic algorithm to search for Liouville solutions of the differential equation (5.8).

5.2. Existence of Liouville solutions. In the general case, the function $\Pi(x)$ has four finite second-order poles at $x=0$, $x=1$, $x=x_1$, and $x=x_2$. We assume that all these poles are distinct. The Laurent expansion of $\Pi(x)$ at $x=\infty$ has the form

$$\Pi(x)\big|_{x=\infty}\approx-\frac{3}{16x^2}+O\left(\frac{1}{x^3}\right).$$

All initial preparations for application of the Kovacic algorithm have been made.

First, we search a solution of the form (2.4) for the differential equation (5.8). Direct application of the Kovacic algorithm to Eq. (5.8) yields the following result.

Theorem 12. *Assume that all poles of the function* $\Pi(x)$ *are distinct. Then Eq.* (5.8) *has a solution of the form* (2.4) *if and only if the Mushtari condition* (5.5) *holds.*

Proof. We apply the Kovacic algorithm for the differential equation (5.8) as described in Sec. 2.3.1.

Step 1. Let us calculate the following values:

$$[\sqrt{\Pi}]_0 = 0, \qquad \alpha_0^+ = \frac{5}{4}, \qquad \alpha_0^- = -\frac{1}{4}, \qquad\qquad [\sqrt{\Pi}]_{x_2} = 0, \qquad \alpha_{x_2}^+ = \frac{3}{4}, \qquad \alpha_{x_2}^- = \frac{1}{4},$$
$$[\sqrt{\Pi}]_1 = 0, \qquad \alpha_1^+ = \frac{3}{2}, \qquad \alpha_1^- = -\frac{1}{2}, \qquad\qquad [\sqrt{\Pi}]_\infty = 0, \qquad \alpha_\infty^+ = \frac{3}{4}, \qquad \alpha_\infty^- = \frac{1}{4}.$$
$$[\sqrt{\Pi}]_{x_1} = 0, \qquad \alpha_{x_1}^+ = \frac{3}{4}, \qquad \alpha_{x_1}^- = \frac{1}{4},$$

Step 2. Since the number ρ of finite poles of the function $\Pi(x)$ is equal to 4, we have $2^{\rho+1} = 2^5 = 32$ tuples of signs

$$s = \big(s(\infty),\ s(0),\ s(1),\ s(x_1),\ s(x_2)\big).$$

For each of these tuples, we calculate d by the formula (2.19):

$$d = \alpha_\infty^{s(\infty)} - \alpha_0^{s(0)} - \alpha_1^{s(1)} - \alpha_{x_1}^{s(x_1)} - \alpha_{x_2}^{s(x_2)}.$$

According to the algorithm, d must be nonnegative integer. Analyzing all possible tuples of signs s and corresponding tuples of α, we conclude that for the tuples

$$p_1 = \big(\alpha_\infty^+,\ \alpha_0^-,\ \alpha_1^-,\ \alpha_{x_1}^+,\ \alpha_{x_2}^+\big) = \left(\frac{3}{4},\ -\frac{1}{4},\ -\frac{1}{2},\ \frac{3}{4},\ \frac{3}{4}\right),$$
$$p_2 = \big(\alpha_\infty^-,\ \alpha_0^-,\ \alpha_1^-,\ \alpha_{x_1}^+,\ \alpha_{x_2}^-\big) = \left(\frac{1}{4},\ -\frac{1}{4},\ -\frac{1}{2},\ \frac{3}{4},\ \frac{1}{4}\right),$$
$$p_3 = \big(\alpha_\infty^-,\ \alpha_0^-,\ \alpha_1^-,\ \alpha_{x_1}^-,\ \alpha_{x_2}^+\big) = \left(\frac{1}{4},\ -\frac{1}{4},\ -\frac{1}{2},\ \frac{1}{4},\ \frac{3}{4}\right)$$

we have $d = 0$, and for the tuple

$$p_4 = \big(\alpha_\infty^+,\ \alpha_0^-,\ \alpha_1^-,\ \alpha_{x_1}^-,\ \alpha_{x_2}^-\big) = \left(\frac{3}{4},\ -\frac{1}{4},\ -\frac{1}{2},\ \frac{1}{4},\ \frac{1}{4}\right)$$

we have $d = 1$.

Consider the tuple p_1. The corresponding function $\theta = \theta(x)$ defined by (2.20) for the chosen tuple of values α has the form

$$\theta = -\frac{1}{4x} - \frac{1}{2(x-1)} + \frac{3}{4(x-x_1)} + \frac{3}{4(x-x_2)}.$$

Step 3. For the tuple p_1 obtained on the previous step, we search for a polynomial of degree $d = 0$ that satisfy the differential equation (2.21). Substituting $P \equiv 1$ in (2.21), we obtain

$$\frac{(2x_1x_2 - x_1 - x_2)(1+x)}{4x(x-1)(x-x_1)(x-x_2)} = 0.$$

This equality holds if the parameters x_1 and x_2 satisfy the condition

$$2x_1x_2 - x_1 - x_2 = 0. \tag{5.9}$$

If we express the condition (5.9) through the initial parameters, we obtain

$$\frac{2m\lambda^2(2A_1 - A_3)}{A_1A_3 + 4m\lambda^2(A_3 - A_1)} = 0.$$

Thus, the condition (5.9) is equivalent to the Mushtari condition (5.5).

One can verify that the conditions for x_1 and x_2 for the remaining tuples p_2, p_3, and p_4 are not physically admissible. Thus we proved that Eq. (5.8) has a solution of the form (2.4) only when the Mushtari condition is valid. Theorem is proved. □

Now we search for a solution of the form (2.9) for the differential equation (5.8), i.e., the solution described in Case 2 of Theorem 1. The necessary conditions of existence of a such solution are fulfilled (see Theorem 4). Direct application of the Kovacic algorithm to the differential equation (5.8) yields the following result.

Theorem 13. *If all poles of the function* $\Pi(x)$ *are distinct, then all solutions of the differential equation* (5.8) *are Liouville solutions and have the form* (2.9).

Proof. We apply the Kovacic algorithm as described in Sec. 2.3.3.

Step 1. Let us define the following sets of integers:

$$E_1 = \{-2, 2, 6\}, \quad E_0 = \{-1, 2, 5\}, \quad E_{x_1} = \{1, 2, 3\}, \quad E_{x_2} = \{1, 2, 3\}, \quad E_\infty = \{1, 2, 3\}.$$

Step 2. Now we consider all possible tuples

$$s = \big(e_\infty,\ e_1,\ e_0,\ e_{x_1},\ e_{x_2}\big)$$

of elements from E_∞, E_1, E_0, E_{x_1}, and E_{x_2}, where each tuple s must contain at least one odd number. We calculate d for each tuple s by (2.28):

$$d = \frac{1}{2}\big(e_\infty - e_1 - e_0 - e_{x_1} - e_{x_2}\big).$$

According to the algorithm, d must be nonnegative integer. Among the tuples of elements from E_∞, E_1, E_0, E_{x_1}, and E_{x_2}, where d is a nonnegative integer, we choose the tuple

$$e = \big(e_\infty,\ e_1,\ e_0,\ e_{x_1},\ e_{x_2}\big) = \big(3,\ -2,\ -1,\ 1,\ 1\big),$$

for which $d = 2$.

Step 3. By the formula (2.29), we construct the function θ using the tuple e obtained in the previous step:

$$\theta = -\frac{1}{x-1} - \frac{1}{2x} + \frac{1}{2(x-x_1)} + \frac{1}{2(x-x_2)}.$$

A polynomial of degree $d = 2$ $P = x^2 + k_1x + k_0$ must satisfy Eq. (2.30) identically. Substituting the function θ and the polynomial P in Eq. (2.30) we obtain

$$k_0 = \frac{x_1x_2}{1 - 2x_1 - 2x_2 + 4x_1x_2}, \quad k_1 = -\frac{2x_1x_2}{1 - 2x_1 - 2x_2 + 4x_1x_2},$$

i.e., the polynomial P exists for all values of x_1 and x_2 and has the form

$$P = x^2 - \frac{2x_1x_2}{1 - 2x_1 - 2x_2 + 4x_1x_2}x + \frac{x_1x_2}{1 - 2x_1 - 2x_2 + 4x_1x_2}.$$

Further, according to the algorithm, we introduce the function

$$\varphi = \theta + \frac{P'}{P} = \frac{1}{2(x-x_1)} + \frac{1}{2(x-x_2)} - \frac{1}{x-1} - \frac{1}{2x} + \frac{2(1-2x_1-2x_2+4x_1x_2)x - 2x_1x_2}{(1-2x_1-2x_2+4x_1x_2)x^2 - 2x_1x_2x + x_1x_2}$$

and find the function ω, which is a solution of the quadratic equation (2.31)

$$\omega = f(x) \pm ig(x),$$

where

$$f(x) = \frac{\varphi}{2} = \frac{1}{4(x-x_1)} + \frac{1}{4(x-x_2)} - \frac{1}{2(x-1)} - \frac{1}{4x} + \frac{(1-2x_1-2x_2+4x_1x_2)x - x_1x_2}{(1-2x_1-2x_2+4x_1x_2)x^2 - 2x_1x_2x + x_1x_2},$$

$$g(x) = \frac{D(1-x)}{(1-2x_1-2x_2+4x_1x_2)x^2 - 2x_1x_2x + x_1x_2}\sqrt{\frac{x}{(x-x_1)(x-x_2)}},$$
$$D = \sqrt{x_1x_2(2x_1-1)(2x_2-1)(2x_1x_2-x_1-x_2)}.$$

Thus, according to the algorithm, the solution of differential equation (5.8) has the form

$$y(x) = \exp\left(\int f(x)dx\right)\left(c_1\cos\left(\int g(x)dx\right) + c_2\sin\left(\int g(x)dx\right)\right),$$

where c_1 and c_2 are arbitrary constants. The theorem is proved. □

Now we obtain the solution of the initial differential equation (5.4). We perform the change of variable

$$r(x) = y(x)\exp\left(-\frac{1}{2}\int d_1(x)dx\right),$$

where $d_1(x)$ is a coefficient of dr/dx in Eq. (5.6). In the obtained expression for $r(x)$, we turn to the initial parameters m, λ, A_1, and A_3 using (5.7). Then we set $x = \cos^2\theta$. Finally, the solution of Eq. (5.4) has the form

$$r(\theta) = \sqrt{\frac{K_1(\theta)}{K_2(\theta)}}(c_1\cos\Phi(\theta) + c_2\sin\Phi(\theta)), \tag{5.10}$$

where

$$\Phi(\theta) = 2m\lambda^2 D\int_0^\theta \frac{\sin^3\varphi\cos^2\varphi d\varphi}{K_1(\varphi)\sqrt{K_2(\varphi)}}, \quad D = \sqrt{2A_1A_3(A_3+4m\lambda^2)(2A_1-A_3)},$$
$$K_1(\theta) = (A_1A_3 + 4A_1m\lambda^2)\cos^4\theta - 2A_3m\lambda^2\cos^2\theta + A_3m\lambda^2,$$
$$K_2(\theta) = (A_1A_3 + 4m\lambda^2(A_3-A_1))\cos^4\theta - 4m\lambda^2(A_3-A_1)\cos^2\theta + A_3m\lambda^2.$$

Note that the formula (5.10) includes the particular case studied by Kh. M. Mushtari (see [35]). Indeed, if we set $A_3 = 2A_1$ in (5.10), we obtain $r = r_0 = \text{const}$, i.e., the Mushtari solution. Combining

(3.24) or (3.29) with (5.10), one can obtain the explicit form for $p(\theta)$:

$$p(\theta) = -\frac{D\cos^2\theta}{2A_1\sin\theta\sqrt{K_1(\theta)}}\Big(c_2\cos\Phi(\theta) - c_1\sin\Phi(\theta)\Big)$$
$$-\frac{A_3\big((A_3+4m\lambda^2)\cos^4\theta - 4m\lambda^2\cos^2\theta + 2m\lambda^2\big)\cos\theta}{\sqrt{K_1(\theta)K_2(\theta)}\sin\theta}\Big(c_1\cos\Phi(\theta) + c_2\sin\Phi(\theta)\Big). \quad (5.11)$$

Thus, the general solution of the system (3.24) in the problem of motion of a rotationally symmetric paraboloid on a perfectly rough horizontal plane is determined by the formulas (5.10) and (5.11). Note that these formulas have been obtained firstly by A.S. Kuleshov (see [23]) without application of the Kovacic algorithm. It can be proved that the function $\Phi(\theta)$ is expressed in terms of elliptic integrals that cannot be simplified if the Mushtari condition (5.5) for the moments of inertia does not hold. Therefore, investigation of motion of the rotationally symmetric paraboloid on the fixed perfectly rough horizontal plane is similar for all values of parameters except for the case (5.5). In the next section, a qualitative analysis of the motion of the paraboloid is considered for the case of a homogeneous segment of the paraboloid whose center of mass coincides with the focus of the generating parabola.

5.3. Motion of a homogeneous segment of the paraboloid.

5.3.1. Evolution of the angle θ. Consider a homogeneous, rotationally symmetric segment of the paraboloid rolling on a perfectly rough horizontal plane (the paraboloidal segment). Let the center of mass of the segment coincide with the focus of the generating parabola with the parameter 2λ. The plane normal to the symmetry axis of the paraboloid bounds its height, and the height is uniquely determined by the position of the center of mass. One can show that the equation of this plane in the coordinate system $G\xi\eta\zeta$ with origin located at the center of mass of the segment has the form $\zeta = \lambda/2$. The moments of inertia of the segment with respect to the central principal axes are

$$A_1 = \frac{9}{8}m\lambda^2, \quad A_3 = 2m\lambda^2. \quad (5.12)$$

We assume that during the motion the segment contacts with the horizontal plane only by the parabolic part of its surface. Then the range of the angle θ is

$$-\theta_* \le \theta \le \theta_*, \quad \theta_* = \arccos\sqrt{\frac{2}{5}}. \quad (5.13)$$

Using the relations (5.12) for the moments of inertia A_1 and A_3, we obtain that the functions $r(\theta)$ and $p(\theta)$ defined by (5.10) and (5.11) have the form

$$r(\theta) = \sqrt{\frac{27\cos^4\theta - 16\cos^2\theta + 8}{23\cos^4\theta - 14\cos^2\theta + 8}}\Big(c_1\cos\Phi(\theta) + c_2\sin\Phi(\theta)\Big),$$
$$p(\theta) = -\frac{4\cos\theta}{\sin\theta\sqrt{27\cos^4\theta - 16\cos^2\theta + 8}}\Bigg[\frac{\sqrt{3}\cos\theta}{3}\Big(c_2\cos\Phi(\theta) - c_1\sin\Phi(\theta)\Big)$$
$$+\frac{4(3\cos^4\theta - 2\cos^2\theta + 1)}{\sqrt{23\cos^4\theta - 14\cos^2\theta + 8}}\Big(c_1\cos\Phi(\theta) + c_2\sin\Phi(\theta)\Big)\Bigg], \quad (5.14)$$
$$\Phi(\theta) = \int_0^\theta \frac{24\sqrt{3}\sin^3\varphi\cos^2\varphi\, d\varphi}{(27\cos^4\varphi - 16\cos^2\varphi + 8)\sqrt{23\cos^4\varphi - 14\cos^2\varphi + 8}}.$$

The energy integral (3.23) can be rewritten as follows:

$$A_1p^2 + (A_1 + m(\xi^2 + \zeta^2))q^2 + A_3r^2 + m(p\zeta - r\xi)^2 + 2mgf(\theta) = c_0^2 = \text{const}, \quad (5.15)$$

Combining (5.12) with (5.15), we have

$$\frac{m\lambda^2(8+9\cos^4\theta)}{8\cos^4\theta}q^2 = c_0^2 - \frac{2mg\lambda}{\cos\theta} - m\lambda^2\left[\frac{9}{8}p^2 + 2r^2 + \left(\frac{2\sin\theta}{\cos\theta}r + \frac{(1-2\cos^2\theta)}{\cos^2\theta}p\right)^2\right].$$

Multiplying both sides by $\sin^2\theta$ and combining it with (3.22), we obtain

$$\frac{m\lambda^2(9+8u^4)}{8u^3}\left(\frac{du}{dt}\right)^2 = F(u) = \frac{(u^2-1)(c_0^2-2mg\lambda u)}{u} - K_0, \tag{5.16}$$

where $u = 1/\cos\theta$, and the function K_0 is determined by

$$K_0 = \frac{m\lambda^2}{u}\left[\frac{9}{8}u^2p_1^2 + 2(u^2-1)r^2 + (2(u^2-1)r + u(u^2-2)p_1)^2\right], \quad p_1 = p\sin\theta.$$

From the definition of u and (5.13) it follows that

$$1 \le u \le \sqrt{\frac{5}{2}}. \tag{5.17}$$

Note that the left-hand side of Eq. (5.16) is nonnegative. Hence the inequality $F(u) \ge 0$ determines the set U of possible values of u. From (5.17) it follows that if $u > \sqrt{5/2}$, then motion is impossible; therefore, in real motion

$$F(u) \ge 0, \quad u \in U, \quad F\left(\sqrt{\frac{5}{2}}\right) \le 0, \tag{5.18}$$

where U is a subset of the interval (5.17). If $c_1 = c_2 = 0$, then $p = r = 0$ for all time. The symmetry axis $G\zeta$ of the segment moves in a fixed vertical plane. The time dependence of the angle θ between the $G\zeta$-axis and the vertical is determined by (5.16), where we set $K_0 = 0$:

$$\frac{m\lambda^2}{8}(9+8u^4)\left(\frac{du}{dt}\right)^2 = u^2(u^2-1)(c_0^2-2mg\lambda u).$$

From (5.17) and (5.18) it follows that in the considered particular case the paraboloidal segment moves with a limited energy range:

$$2mg\lambda \le c_0^2 \le \sqrt{10}mg\lambda.$$

Therefore, if $c_0^2 < 2mg\lambda$, motion is impossible. The case $c_0^2 = 2mg\lambda$ corresponds to the equilibrium of the segment. Moreover, $u = 1$ and the center of mass is situated in the lowest position. If

$$2mg\lambda < c_0^2 \le \sqrt{10}mg\lambda,$$

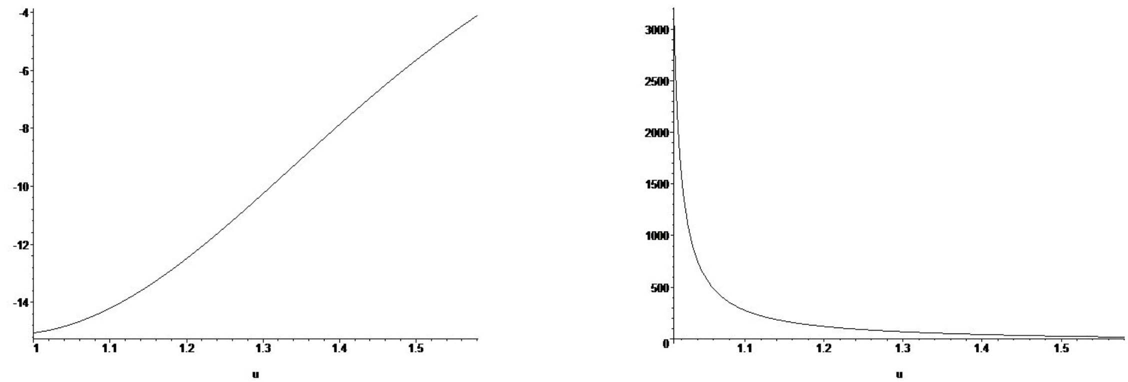
then the symmetry axis of the segment oscillates in a fixed vertical plane with amplitude limited by θ_* and the track of the point of contact on the supporting plane is a segment of a straight line.

Further, let us examine the motion of the homogeneous paraboloidal segment in general case. First, we prove the following property of the function $F(u)$.

Proposition 1. The function $F(u)$ is concave upward on the interval (5.17).

Proof. Let us transform the system (5.3) using formulas (5.12) for the moments of inertia A_1 and A_3. Then we perform the change of the independent variable by the formula $1/\cos\theta = u$. As a result, we obtain

$$\begin{cases} \dfrac{dp_1}{du} = -\dfrac{16}{u^2(8u^4-14u^2+23)}\Big((u^5-3u^3+2u)p_1 + \big(u^4-4u^2+1\big)r\Big), \\ \dfrac{dr}{du} = -\dfrac{2}{u(8u^4-14u^2+23)}\Big(9u\big(u^2-1\big)p_1 + \big(17u^2-16\big)r\Big). \end{cases} \tag{5.19}$$



(a) (b)

Fig. 6. The graphs of the functions (a) $\dfrac{q_{11}}{m\lambda^2}$; (b) $\dfrac{q_{11}q_{22}-q_{12}^2}{m^2\lambda^4}$.

The energy integral (5.15) can be transformed to the form

$$\frac{m\lambda^2u^2(9+8(u^2-2)^2)}{8(u^2-1)}p_1^2+\frac{1}{8}m\lambda^2(9+8u^4)q^2+2m\lambda^2(2u^2-1)r^2 \\ +4m\lambda^2u(u^2-2)p_1r+2mg\lambda u=c_0^2. \quad (5.20)$$

We differentiate the function $F(u)$ twice and substitute the right-hand sides of (5.19) for the derivatives after every differentiation. Then we exclude c_0^2 from the obtained expression using (5.20). As a result, we obtain the following expression for the second derivative of the function $F(u)$:

$$F''(u)=q_0+q_{11}r^2+2q_{12}p_1r+q_{22}p_1^2,$$

where

$$q_0=-\frac{4mg\lambda(u^2+1)}{u^2}-\frac{m\lambda^2(9+8u^4)}{4u^3}q^2, \qquad q_{11}=\frac{64m\lambda^2(u^2+1)(u^4-4u^2+1)}{u^3(8u^4-14u^2+23)},$$

$$q_{12}=\frac{4m\lambda^2(u^2+1)(8u^4-41u^2+32)}{u^2(8u^4-14u^2+23)}, \qquad q_{22}=\frac{m\lambda^2(64u^8-528u^6+624u^4+46u^2-495)}{4u(u^2-1)(8u^4-14u^2+23)}.$$

We can show that

$$q_{11}<0, \quad q_{11}q_{22}-q_{12}^2>0$$

on the interval (5.17). The graphs of the functions $\dfrac{q_{11}}{m\lambda^2}$ and $\dfrac{q_{11}q_{22}-q_{12}^2}{m^2\lambda^4}$ shown in Fig. 6 confirm this conclusion.

Thus, according to the Sylvester criterion (see [12]), the quadratic form

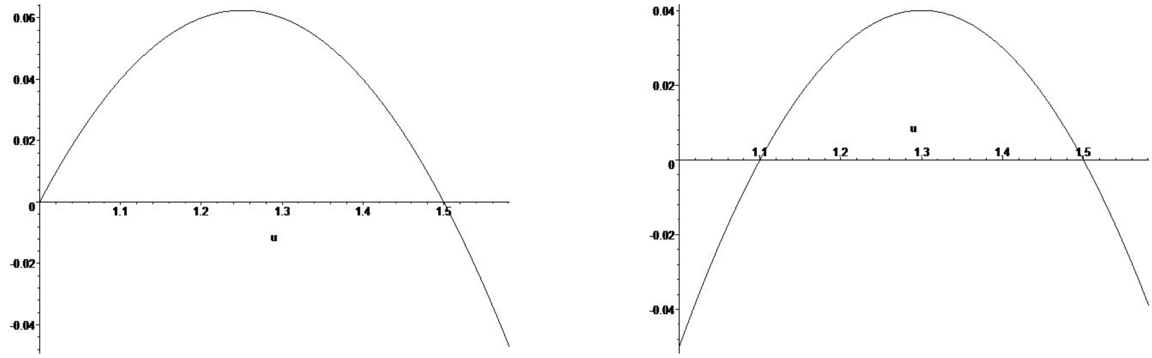
$$Q(r,p_1)=q_{11}r^2+2q_{12}p_1r+q_{22}p_1^2$$

is negative definite for all values of u that satisfy (5.17). Obviously, $q_0<0$ on the same interval. Therefore, $F''(u)<0$ and the function $F(u)$ is concave upward on the considered interval. The proposition is proved. □

Note that

$$F(1)=-\frac{2m\lambda^2}{171}\big(24c_1+\sqrt{51}c_2\big)^2\le 0. \quad (5.21)$$

Taking into account the fact that the function $F(u)$ is concave upward and using (5.18) and (5.21), we conclude that this function has two zeros u_1 and u_2 on the interval (5.17). These zeros are boundary



(a) (b)

Fig. 7. Possible types of graphs of the function $F(u)$: (a) $F(1) = 0$; (b) $F(1) < 0$.

points of the set U. Hence, U is the interval $[u_1, u_2]$. Therefore, $u(t) = 1/\cos\theta(t)$ is a periodic time-dependent function and u varies between u_1 and u_2, i.e., $u_1 \le u \le u_2$. According to (5.21), if $24c_1 + \sqrt{51}c_2 = 0$, then we have $F(1) = 0$. Otherwise, $F(1) < 0$. Thus, two types of graphs of the function $F(u)$ are possible. They correspond to various different behaviour of this function at $u = 1$ (see Fig. 7).

The cases $F(1) = 0$ and $F(1) < 0$ are dynamically distinct. If $F(1) = 0$, then the point $u = 1$ belongs to U (it is a boundary point u_1). Since $u = 1/\cos\theta$, the value $u = 1$ corresponds to $\theta = 0$. This means that during the motion the paraboloidal segment periodically takes the position when its axis of symmetry is vertical. In this motion the angle θ varies between symmetric limits:

$$-\theta_0 \le \theta \le \theta_0, \quad \theta_0 = \arccos\frac{1}{u_2}.$$

If $F(1) < 0$, then $u > 1$ and, therefore, θ is a function of fixed sign for all time. The range of angle θ is nonsymmetric:

$$-\theta_2 \le \theta \le -\theta_1 < 0 \quad \text{or} \quad 0 < \theta_1 \le \theta \le \theta_2, \qquad \theta_i = \arccos\frac{1}{u_i}, \quad i = 1, 2,$$

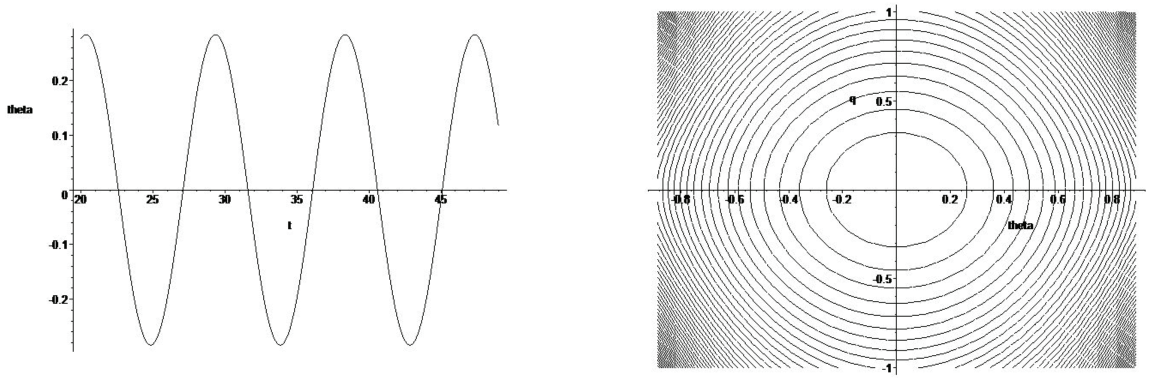
and the segment is permanently inclined to the vertical. The angle of inclination is a periodic function of time.

5.3.2. Case $F(1) = 0$. From (5.21) we obtain

$$c_2 = -\frac{24}{\sqrt{51}}c_1. \tag{5.22}$$

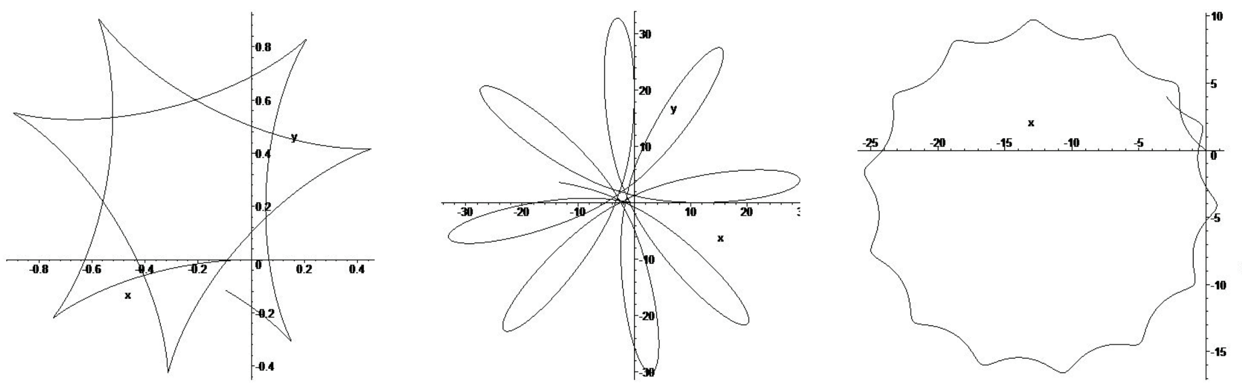
Now we substitute the obtained expression (5.22) for c_2 to the solution (5.14) and get

$$\begin{aligned} r &= c_1\sqrt{\frac{27\cos^4\theta - 16\cos^2\theta + 8}{23\cos^4\theta - 14\cos^2\theta + 8}}\left(\cos\Phi(\theta) - \frac{24}{\sqrt{51}}\sin\Phi(\theta)\right), \\ p &= \frac{4c_1\cos\theta}{\sin\theta\sqrt{27\cos^4\theta - 16\cos^2\theta + 8}}\left[\left(\frac{8}{\sqrt{17}}\cos\Phi(\theta) + \frac{\sqrt{3}}{3}\sin\Phi(\theta)\right)\cos\theta\right. \\ &\qquad \left. - \frac{4\left(3\cos^4\theta - 2\cos^2\theta + 1\right)}{\sqrt{23\cos^4\theta - 14\cos^2\theta + 8}}\left(\cos\Phi(\theta) - \frac{24}{\sqrt{51}}\sin\Phi(\theta)\right)\right]. \end{aligned} \tag{5.23}$$



(a) (b)

Fig. 8. (a) The graph of the function $\theta(t)$ for $g = \lambda = 1$ and $c_1 = 0.1$. (b) The phase portrait in the plane $(\theta, \dot{\theta})$.



(a) (b) (c)

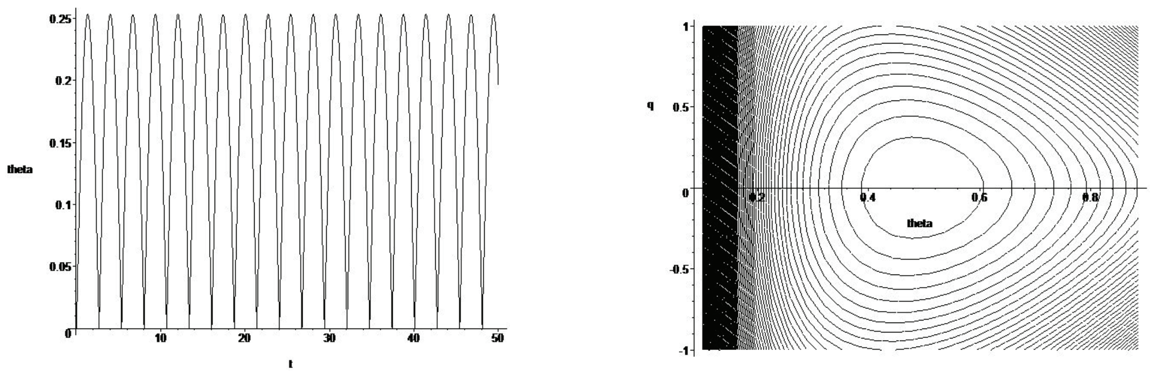
Fig. 9. The trajectory of the contact point on the horizontal plane for (a) $c_1 = 0.3$, $\theta(0) = 0.1$, $q(0) = 0.2$, $\Phi(0) = 0$; (b) $c_1 = 0.1$, $\theta(0) = 0.8$, $q(0) = 1.2$, $\Phi(0) = \pi/4$; (c) $c_1 = 1$, $\theta(0) = 0.8$, $q(0) = 0.3$, $\Phi(0) = \pi$.

Then, using (5.23), we study the remaining equations of motion of the paraboloidal segment. The first equation of the system (3.21) can be written as follows:

$$\frac{dq}{dt} = \frac{8}{8 + 9\cos^4\theta}\left(\frac{g}{\lambda}\sin\theta\cos^2\theta - \frac{\cos\theta}{8\sin\theta}\left(9\cos^4\theta + 16\cos^2\theta - 8\right)p^2 \right.$$
$$\left. + 2\cos^2\theta(1 + \cos^2\theta)pr + \frac{2\sin\theta}{\cos\theta}q^2\right), \quad (5.24)$$

and Eq. (3.22) in the form $d\theta/dt = -q$. Using (5.14), we have the following differential equation for the function $\Phi(\theta)$:

$$\frac{d\Phi}{dt} = -\frac{24\sqrt{3}q\sin^3\theta\cos^2\theta}{(27\cos^4\theta - 16\cos^2\theta + 8)\sqrt{23\cos^4\theta - 14\cos^2\theta + 8}}. \quad (5.25)$$



(a) (b)

Fig. 10. (a) The graph of the function $\theta(t)$ for $g = \lambda = 1$. (b) The phase portrait on the plane $(\theta, \dot{\theta})$.

Now to Eqs. (5.24) and (5.25) we add Eqs. (3.25):

$$\frac{d\varphi}{dt} = r - p\cot\theta, \quad \frac{d\psi}{dt} = \frac{p}{\sin\theta}. \tag{5.26}$$

From (3.28), using (3.25), one can derive two differential equations for evolution of the coordinates x and y of the point M of contact of the paraboloidal segment and the horizontal plane:

$$\begin{cases} \dfrac{dx}{dt} = -2\lambda p \sin\psi - \dfrac{2\lambda\cos\psi}{\cos^3\theta} q + \dfrac{2\lambda\sin\theta\sin\psi}{\cos\theta} r, \\ \dfrac{dy}{dt} = 2\lambda p\cos\psi - \dfrac{2\lambda\sin\psi}{\cos^3\theta} q - \dfrac{2\lambda\sin\theta\cos\psi}{\cos\theta} r. \end{cases} \tag{5.27}$$

We substitute the expressions for p and r defined by (5.23) in Eqs. (5.24), (5.26), and (5.27). Then we obtain the complete system of differential equations (5.24)–(5.27) with respect to variables $q(t)$, $\Phi(t)$, $\theta(t)$, $\psi(t)$, $\varphi(t)$, $x(t)$, and $y(t)$. Thus, the Euler angles θ, ψ, and φ and the coordinates x and y of the contact point M of the paraboloidal segment with the horizontal plane can be found from system (5.24)–(5.27) as functions of time t. The mentioned system of differential equations was solved numerically by the Runge–Kutta–Fehlberg method of order 4. Note that all singularities for $\theta = 0$ in the system of equations are removable. Figure 8 displays the dependence $\theta(t)$ and the corresponding phase portrait in the plane $(\theta, \dot{\theta})$. Figure 9 illustrates typical trajectories of the contact point M on the horizontal plane for different values of c_1 and fixed values of g and λ.

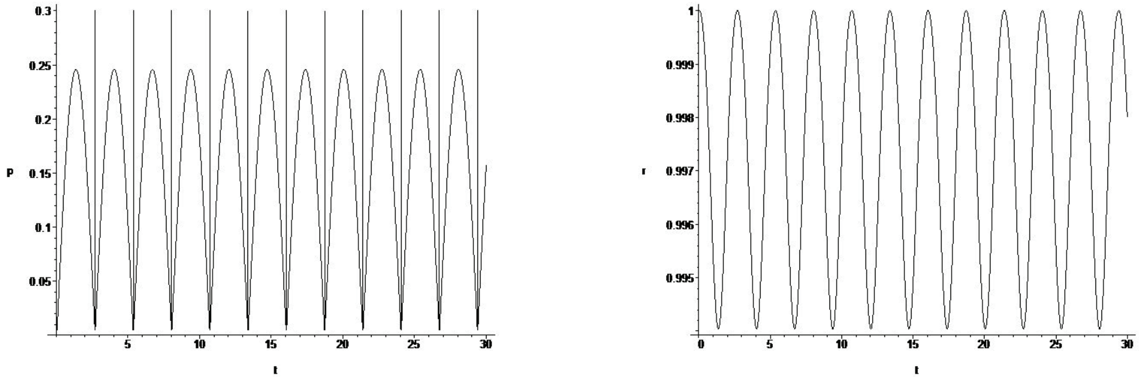
5.3.3. Case $F(1) < 0$. From the system (5.24)–(5.27), we exclude Eq. (5.25) for the function $\Phi(t)$ and complete the system (5.24), (5.26), (5.27) by the equations

$$\begin{cases} \dfrac{dp}{dt} = -\dfrac{(9\cos^6\theta - 66\cos^4\theta + 56\cos^2\theta - 16)pq}{(23\cos^4\theta - 14\cos^2\theta + 8)\sin\theta\cos\theta} + \dfrac{16(\cos^4\theta - 4\cos^2\theta + 1)qr}{23\cos^4\theta - 14\cos^2\theta + 8}, \\ \dfrac{dr}{dt} = \dfrac{18\sin^4\theta pq}{23\cos^4\theta - 14\cos^2\theta + 8} - \dfrac{2(16\cos^2\theta - 17)\sin\theta\cos\theta qr}{23\cos^4\theta - 14\cos^2\theta + 8}, \end{cases} \tag{5.28}$$

which can be derived from the system (5.3). In addition to these equations, we find the coordinates ξ and ζ as functions of time using the equations:

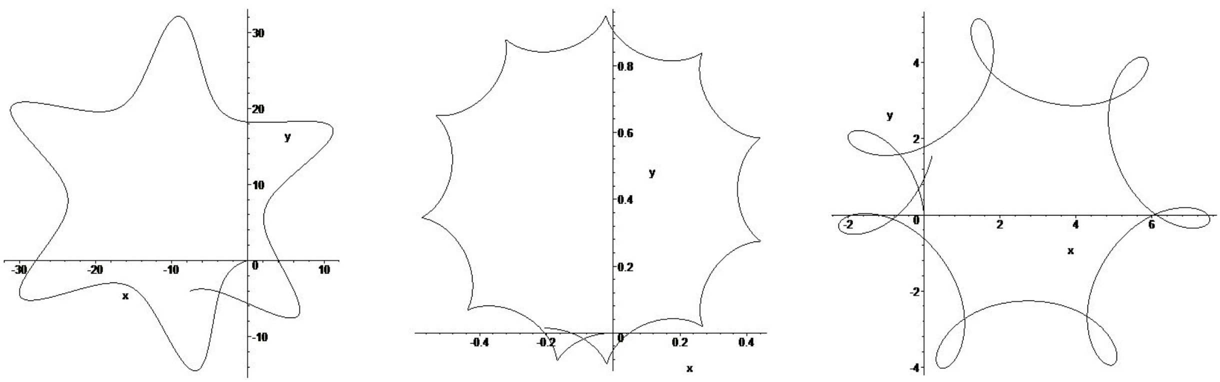
$$\frac{d\xi}{dt} = \frac{2\lambda q}{\cos^2\theta}, \quad \frac{d\zeta}{dt} = -\frac{2\lambda q\sin\theta}{\cos^3\theta}. \tag{5.29}$$

These equations can be derived from (5.2) by differentiation with respect to time.



(a) (b)

Fig. 11. The graphs of the functions (a) $p(t)$, (b) $r(t)$ for $g = \lambda = 1$.



(a) (b) (c)

Fig. 12. The trajectory of the contact point on the supporting plane for (a) $\theta(0) = 0.8$, $p(0) = 0.1$, $q(0) = 1$, $r(0) = 1$; (b) $\theta(0) = 0.01$, $p(0) = 0.01$, $q(0) = 0.3$, $r(0) = 1$; (c) $\theta(0) = 0.1$, $p(0) = 1$, $q(0) = 0.1$, $r(0) = 0.1$.

As a result, Eqs. (5.24) and (5.26)–(5.29) form a complete system of differential equations with respect to the unknown functions $p(t)$, $q(t)$, $r(t)$, $\theta(t)$, $\psi(t)$, $\varphi(t)$, $x(t)$, $y(t)$, $\xi(t)$, and $\zeta(t)$. Fixing constants g and λ and initial conditions for all listed functions, we can solve the system of equations numerically. The Runge–Kutta–Fehlberg method of order 4 has been applied for integration. Figure 10 shows the graph of the function $\theta(t)$ and the phase portrait in the plane $(\theta, \dot{\theta})$. In Fig. 11 the graphs of components $p(t)$ and $r(t)$ of angular velocity are shown. Figure 12 illustrates typical trajectories of the contact point M on the horizontal plane.

The above analysis shows that the trajectory of the contact point M on the surface of the paraboloidal segment is a curve constructed from periodically repeated waves touching alternately two parallels of the paraboloid. The trajectory of the point of contact on the horizontal plane is a similar curve bounded between two concentric circles that are touched by the point of contact alternately while the paraboloid moves on the plane. The motion of the paraboloidal segment is quasiperiodic. Our conclusions are consistent with the results previously obtained by N. K. Moshchuk (see [33, 34]).

5.4. Steady Motions of the Paraboloid and Their Stability. In this section, we study the existence and stability of steady motions of the paraboloid (see [16]). From (5.1) we have

$$f'(\theta)\Big|_{\theta=0} = \frac{\lambda \sin\theta}{\cos^2\theta}\Big|_{\theta=0} = 0,$$

i.e., the symmetry axis $G\zeta$ of the paraboloid intersects its surface being normal to it for negative values of ζ. In this case, the paraboloid can move in a such way that (see [31])

$$p = 0, \quad r = \omega = \text{const}, \quad q = 0, \quad \theta = 0. \tag{5.30}$$

In this motion, the paraboloid rotates about its axis of symmetry, which is fixed and vertical, with an arbitrary constant angular velocity ω. The stability condition of the steady motion (5.30) is

$$\Big(A_3 + mf_0\big(f_0 + f_0''\big)\Big)^2\omega^2 + 4mgf_0''\big(A_1 + mf_0^2\big) > 0 \tag{5.31}$$

(see [31]), where the subscript 0 denotes the values of function $f(\theta)$ and its second derivative for $\theta = 0$. In the case where the center of mass of the paraboloid is situated at the focus of the generating parabola, we have

$$f_0'' = f''(\theta)\big|_{\theta=0} = \frac{\lambda(1+\sin^2\theta)}{\cos^3\theta}\Big|_{\theta=0} = \lambda > 0;$$

and hence the expression on the left-hand side of (5.31) is always positive; therefore, the solution (5.30) is stable for all values of ω. Further, there exist steady motions of the paraboloid such that the angle θ between the symmetry axis and the vertical remains constant and nonzero (see [16, 31, 32]):

$$\theta = \theta_0 \neq 0, \quad q = 0, \quad p = p_0 \neq 0, \quad r = r_0, \tag{5.32}$$

if the constants θ_0, p_0, and r_0 satisfy the equation

$$a_{11}p_0^2 + a_{12}p_0r_0 - mgf_0' = 0 \tag{5.33}$$

(see [16, 31, 32]), where

$$a_{11} = \left(A_1 - \frac{m\zeta_0}{\cos\theta_0}f_0\right)\cot\theta_0, \quad a_{12} = -\left(A_3 - \frac{m\xi_0}{\sin\theta_0}f_0\right).$$

Here and below, the subscript 0 means that the value of the corresponding function is calculated at $\theta = \theta_0$.

Let us consider Eq. (5.33) as a quadratic equation with respect to p_0 and require that its roots be real. Then we obtain the following condition of the existence of the solution (5.32):

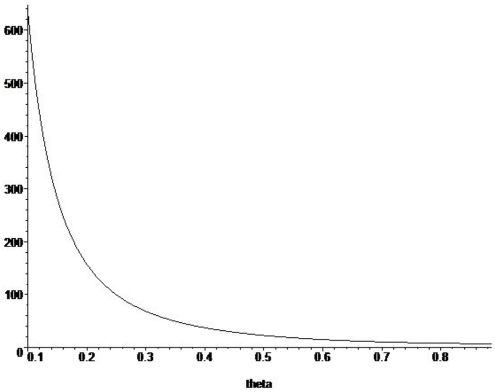
$$\left(A_3 - \frac{m\xi_0}{\sin\theta_0}f_0\right)^2 r_0^2 + 4mgf_0'\left(A_1 - \frac{m\zeta_0}{\cos\theta_0}f_0\right)\cot\theta_0 \geq 0.$$

The solution (5.32) corresponds to a regular precession of the paraboloid. The stability condition for the solution (5.32) has the following form (see [16, 31, 32]):

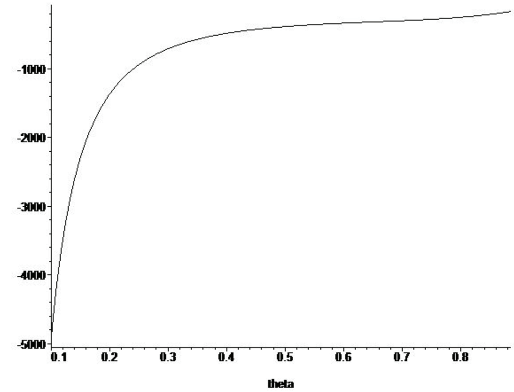
$$b_{11}p_0^2 + b_{12}p_0r_0 + b_{22}r_0^2 + mgf_0'' > 0, \tag{5.34}$$

where

$$b_{11} = \frac{\big(A_1 + m\zeta_0^2\big)\big(1 + 2\cos^2\theta_0\big)}{\sin^2\theta_0} + \frac{m\xi_0\big(\xi_0\sin\theta_0 + 3\zeta_0\cos\theta_0\big)}{\sin\theta_0} + \frac{A_3 m\zeta_0(\xi_0 + \zeta_0')\big((A_1 + m\zeta_0^2)\cos\theta_0 + m\xi_0\zeta_0\sin\theta_0\big)}{\big(A_1A_3 + A_1m\xi_0^2 + A_3m\zeta_0^2\big)\sin\theta_0},$$



(a)



(b)

Fig. 13. The graphs of the functions (a) $b_{11}(\theta_0)/(m\lambda^2)$; (b) $D(\theta_0)/(m^2\lambda^4)$.

$$
\begin{aligned}
b_{12} = -\Big(3A_3 + 3m\xi_0^2 + m\xi_0'\zeta_0\Big)\frac{\cos\theta_0}{\sin\theta_0} - \frac{m\xi_0\zeta_0(1+\cos^2\theta_0)}{\sin^2\theta_0} \\
+ \frac{m\xi_0\big(2A_3 + 2m\xi_0^2 + m\xi_0'\zeta_0\big)}{A_1A_3 + A_1m\xi_0^2 + A_3m\zeta_0^2}\left(A_1\xi_0\frac{\cos\theta_0}{\sin\theta_0} - A_3\zeta_0\right) \\
- \frac{A_3m\zeta_0\zeta_0'}{A_1A_3 + A_1m\xi_0^2 + A_3m\zeta_0^2}\left(A_3 + m\xi_0^2 + \frac{m\xi_0\zeta_0\cos\theta_0}{\sin\theta_0}\right),
\end{aligned}
$$

$$
b_{22} = \left(A_3 + m\xi_0^2 + \frac{m\xi_0\zeta_0\cos\theta_0}{\sin\theta_0}\right)\frac{A_3\big(A_3 + m\xi_0^2 + m\xi_0'\zeta_0\big)}{A_1A_3 + A_1m\xi_0^2 + A_3m\zeta_0^2}.
$$

The expressions of the coefficients b_{ij} in (5.34) are sufficiently complicated. One can show that for a homogeneous paraboloidal segment whose moments of inertia are defined by (5.12) and the angle θ is limited by (5.13), the coefficients b_{ij} can be explicitly expressed as follows:

$$
\begin{aligned}
b_{11} &= \frac{m\lambda^2\big(126\cos^{10}\theta_0 + 1267\cos^8\theta_0 - 430\cos^6\theta_0 - 432\cos^4\theta_0 + 400\cos^2\theta_0 - 64\big)}{8\sin^2\theta_0\cos^4\theta_0\big(23\cos^4\theta_0 - 14\cos^2\theta_0 + 8\big)}, \\
b_{12} &= -\frac{2m\lambda^2\big(1+\cos^2\theta_0\big)\big(5\cos^6\theta_0 + 104\cos^4\theta_0 - 74\cos^2\theta_0 + 16\big)}{\sin\theta_0\cos^3\theta_0\big(23\cos^4\theta_0 - 14\cos^2\theta_0 + 8\big)}, \\
b_{22} &= -\frac{32m\lambda^2\big(1+\cos^2\theta_0\big)\big(\cos^4\theta_0 - 4\cos^2\theta_0 + 1\big)}{\cos^2\theta_0\big(23\cos^4\theta_0 - 14\cos^2\theta_0 + 8\big)};
\end{aligned}
$$

moreover, they satisfy the inequalities

$$
b_{11} > 0, \quad b_{22} > 0, \quad D = b_{12}^2 - 4b_{11}b_{22} < 0. \tag{5.35}
$$

Thus, the quadratic form $b_{11}p_0^2 + b_{12}p_0r_0 + b_{22}r_0^2$ is positive definite for all values of θ_0. Taking into account the fact that the value mgf_0'' is also positive, we conclude that the solution (5.32) is stable whenever it exists. In other words, all regular precessions of the homogeneous paraboloidal segment whose center of mass coincides with the focus of the generating parabola are stable.

In Fig. 13 the graphs of the functions $b_{11}(\theta_0)/(m\lambda^2)$ and $D(\theta_0)/(m^2\lambda^4)$ are shown; these graphs ascertain the validity of the inequalities (5.35).

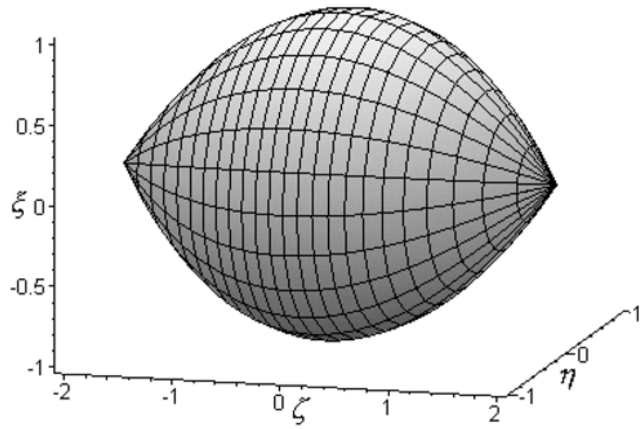


Fig. 14. Motion of a spindle-shaped body on a horizontal plane.

6. Motion of a Spindle-Shaped Body

6.1. Formulation of the problem. Equations of motion. General case and special cases. In this section, we consider the problem of the motion of a so-called spindle-shaped body on a perfectly rough horizontal plane. The surface of this body is formed by rotation of a parabolic arc about the axis passing through the focus and parallel to the directrix. The surface of a such body has two sharp peaks; it looks like a spindle (see Fig. 14). The problem of the motion of a spindle-shaped body was studied by Kh. M. Mushtari (see [35]), who presented a complete solution under the following additional restriction on the moments of inertia of the body:

$$A_3 = \frac{2}{3}A_1. \tag{6.1}$$

We assume that the contact point of the body and the plane lies on the convex surface of the body. The general case where the contact at peaks is also possible was examined by A. A. Zobova (see [37]).

The distance between the center of mass of the body and the horizontal supporting plane is

$$f(\theta) = \frac{\lambda}{\sin\theta}, \quad \lambda = \mathrm{const}\,.$$

Remark 2. Since the contact point lies on the convex surface of the body, the following restriction for θ holds:

$$\theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right).$$

According to (3.11), we find the coordinates ξ and ζ of the contact point:

$$\xi = \frac{\lambda\cos^2\theta}{\sin^2\theta} - \lambda, \quad \zeta = -\frac{2\lambda\cos\theta}{\sin\theta}, \quad \zeta^2 = 4\lambda(\xi + \lambda).$$

Therefore, the system (3.29) can be written as follows:

$$\begin{cases} \dfrac{dp}{d\theta} = \left(\dfrac{2A_3 m\lambda^2\sin^2\theta(3 - 2\sin^2\theta)}{\Delta} - 1\right)\dfrac{\cos\theta}{\sin\theta}p + \dfrac{A_3\big((A_3 + 4m\lambda^2)\sin^4\theta - 8m\lambda^2\sin^2\theta + 5m\lambda^2\big)}{\Delta}r, \\ \dfrac{dr}{d\theta} = \dfrac{A_1 m\lambda^2(1 - 2\sin^2\theta)(3 - 2\sin^2\theta)}{\Delta}p + \dfrac{2m\lambda^2\cos\theta(1 - 2\sin^2\theta)(A_1 - A_3\sin^2\theta)}{\Delta\sin\theta}r, \\ \Delta = \big(A_1A_3 + 4(A_1 - A_3)m\lambda^2\big)\sin^4\theta - 4(A_1 - A_3)m\lambda^2\sin^2\theta + A_1 m\lambda^2, \end{cases}$$

and the differential equation (3.30) takes the form

$$\frac{d^2r}{d\theta^2} + b_1\frac{dr}{d\theta} + b_2 r = 0, \tag{6.2}$$

where

$$b_1 = \frac{\big(4\sin^4\theta - 24\sin^2\theta + 15\big)\cos\theta}{(1-2\sin^2\theta)(3-2\sin^2\theta)\sin\theta} - \frac{6\big(A_1 - 2(A_1 - A_3)\sin^2\theta\big)m\lambda^2\cos\theta}{\Delta\sin\theta},$$

$$b_2 = \frac{(3A_3 - 2A_1)m\lambda^2(1-2\sin^2\theta)^2}{\Delta(3-2\sin^2\theta)}.$$

If the Mushtari condition (6.1) is fulfilled, then the differential equation (6.2) has the following solution (see [35]):

$$r = r_0 = \text{const}\,.$$

In (6.2), we perform the change of the independent variable by the formula $\sin^2\theta = x$ and introduce the notation $B = m\lambda^2$. Then we rewrite Eq. (6.2) as follows:

$$\frac{d^2r}{dx^2} + d_1(x)\frac{dr}{dx} + d_2(x)r = 0, \tag{6.3}$$

where

$$d_1(x) = \frac{18 - 53x + 48x^2 - 12x^3}{2x(1-x)(1-2x)(3-2x)} - \frac{3(A_1 - 2(A_1 - A_3)x)B}{x\Delta},$$

$$d_2(x) = \frac{(3A_3 - 2A_1)(1-2x)^2B}{4x(1-x)(3-2x)\Delta},$$

$$\Delta = (A_1A_3 + 4B(A_1 - A_3))x^2 - 4B(A_1 - A_3)x + A_1B.$$

If $A_1A_3 + 4B(A_1 - A_3) \neq 0$, then the polynomial Δ has two roots x_1 and x_2:

$$\begin{aligned} x_1 &= \frac{2B(A_1 - A_3) - \sqrt{4A_3B^2(A_3 - A_1) - A_1^2A_3B}}{A_1A_3 + 4B(A_1 - A_3)}, \\ x_2 &= \frac{2B(A_1 - A_3) + \sqrt{4A_3B^2(A_3 - A_1) - A_1^2A_3B}}{A_1A_3 + 4B(A_1 - A_3)}. \end{aligned} \tag{6.4}$$

After the change of variable (2.2) in Eq. (6.3) we get

$$\frac{d^2y}{dx^2} = S(x)y, \tag{6.5}$$

where

$$\begin{aligned} S(x) = \frac{\beta_0}{x} + \frac{\beta_1}{x-1} + \frac{\alpha_1}{(x-1)^2} + \frac{\beta_2}{x-1/2} + \frac{\alpha_2}{(x-1/2)^2} \\ + \frac{\beta_3}{x-3/2} + \frac{\alpha_3}{(x-3/2)^2} + \frac{\beta_4}{x-x_1} + \frac{\alpha_4}{(x-x_1)^2} + \frac{\beta_5}{x-x_2} + \frac{\alpha_5}{(x-x_2)^2}, \end{aligned} \tag{6.6}$$

where

$$\alpha_1 = \alpha_4 = \alpha_5 = -\frac{3}{16}, \qquad \alpha_2 = \alpha_3 = \frac{3}{4},$$

$$\beta_0 = \frac{3(x_1 + x_2) - 4x_1x_2}{48x_1x_2}, \qquad \beta_1 = \frac{4x_1x_2 - 9(x_1 + x_2) + 12}{16(x_1 - 1)(x_2 - 1)},$$

$$\beta_2 = \frac{3(x_1 + x_2 - 1)}{(2x_1 - 1)(2x_2 - 1)}, \qquad \beta_3 = \frac{15(x_1 + x_2) - 8x_1x_2 - 27}{3(2x_1 - 3)(2x_2 - 3)},$$

$$\beta_4 = -\frac{(8x_1^3 - 36x_1^2 + 51x_1 - 25)(4x_2 - 3)x_1 + 15(x_1 - 1)x_1 + 3(x_2 - x_1)}{16x_1(x_1 - 1)(2x_1 - 1)(2x_1 - 3)(x_1 - x_2)},$$
$$\beta_5 = \frac{(8x_2^3 - 36x_2^2 + 51x_2 - 25)(4x_1 - 3)x_2 + 15(x_2 - 1)x_2 + 3(x_1 - x_2)}{16x_2(x_2 - 1)(2x_2 - 1)(2x_2 - 3)(x_1 - x_2)}.$$

Thus, the function $S(x)$ has six finite poles: $x = 0$, $x = 1$, $x = 1/2$, $x = 3/2$, $x = x_1$, and $x = x_2$. In the general case, all these poles are distinct. Nevertheless, under some additional conditions for the parameters, the function $S(x)$ has a form different from (6.6). This occurs in the following cases:

1. If
$$A_1 A_3 + 4B(A_1 - A_3) = 0, \tag{6.7}$$
then Δ is a first-degree polynomial and its unique root x_0 is
$$x_0 = \frac{A_1}{4(A_1 - A_3)} = -\frac{m\lambda^2}{A_3}.$$
Since $x_0 < 0$, it does not coincide with the poles $x = 0$, $x = 1$, $x = 1/2$, and $x = 3/2$.
2. If
$$A_1 A_3 + 4B(A_1 - A_3) \neq 0, \quad B = \frac{9A_1 A_3}{4(3A_3 - 4A_1)} \tag{6.8}$$
then $x_1 = 3/2$. It is easy to show that the poles $x = 0$, $x = 1$, and $x = 1/2$ do not coincide with the poles $x = x_1$ and $x = x_2$ for all physically admissible values of parameters.
3. If
$$A_1 A_3 + 4B(A_1 - A_3) \neq 0, \quad B = \frac{A_1^2}{4(A_3 - A_1)} \tag{6.9}$$
we have
$$x_1 = x_2 = \frac{A_1}{2(A_1 - A_3)} < 0.$$
4. Under the condition (6.1), we obtain $\beta_0 = 0$ in (6.6) and, therefore, the function $S(x)$ does not have a first-order pole at $x = 0$. In this case, Eq. (6.5) possesses Liouville solutions (see [35]).

Thus, to perform a complete analysis of the problem on the existence of Liouville solutions of the differential equation (6.5), we must consider the general case where all poles of the function $S(x)$ are distinct and three special cases (6.7), (6.8), and (6.9).

6.2. General case. Assume that all poles of the function $S(x)$ are distinct and the coefficient β_0 is nonzero, i.e., the function $S(x)$ is defined by (6.6). In this case, the Laurent expansion of the function $S(x)$ at $x = \infty$ is
$$S(x)\big|_{x=\infty} \approx -\frac{3}{16x^2} + O\left(\frac{1}{x^3}\right).$$
All initial preparations necessary for the application of the Kovacic algorithm have been performed. Direct application of the algorithm to Eq. (6.5) yields the following result.

Theorem 14. *If the function $S(x)$ is defined by the formula* (6.6)*, then the differential equation* (6.5) *has no Liouville solutions for all physically admissible values of parameters.*

Proof. First, we search for a solution of Eq. (6.5) of the form (2.4), i.e., a solution described in Case 1 of Theorem 1. Note that the function $S(x)$ has five second-order finite poles: $x = 1$, $x = 1/2$, $x = 3/2$, $x = x_1$, and $x = x_2$, the first-order pole at $x = 0$, and a second-order pole at $x = \infty$. Therefore, the necessary conditions for the existence of a solution of the form (2.4) for the differential equation (6.5), are fulfilled (see Theorem 4). Now we apply the Kovacic algorithm as described in Sec. 2.3.1.

Step 1. Let us calculate the following values:

$$[\sqrt{S}]_1 = 0, \qquad \alpha_1^+ = \frac{3}{4}, \qquad \alpha_1^- = \frac{1}{4},$$
$$[\sqrt{S}]_{x_1} = 0, \qquad \alpha_{x_1}^+ = \frac{3}{4}, \qquad \alpha_{x_1}^- = \frac{1}{4},$$
$$[\sqrt{S}]_{x_2} = 0, \qquad \alpha_{x_2}^+ = \frac{3}{4}, \qquad \alpha_{x_2}^- = \frac{1}{4},$$
$$[\sqrt{S}]_{1/2} = 0, \qquad \alpha_{1/2}^+ = \frac{3}{2}, \qquad \alpha_{1/2}^- = -\frac{1}{2},$$
$$[\sqrt{S}]_{3/2} = 0, \qquad \alpha_{3/2}^+ = \frac{3}{2}, \qquad \alpha_{3/2}^- = -\frac{1}{2},$$
$$[\sqrt{S}]_0 = 0, \qquad \alpha_0^+ = 1, \qquad \alpha_0^- = 1,$$
$$[\sqrt{S}]_\infty = 0, \qquad \alpha_\infty^+ = \frac{3}{4}, \qquad \alpha_\infty^- = \frac{1}{4}.$$

Step 2. Since the number ρ of the finite poles of the function $S(x)$ is equal to 6, we have $2^{\rho+1} = 2^7 = 128$ tuples of signs

$$s = \big(s(\infty),\ s(1),\ s(x_1),\ s(x_2),\ s(1/2),\ s(3/2),\ s(0)\big).$$

For each tuple, we calculate d by (2.19):

$$d = \alpha_\infty^{s(\infty)} - \alpha_1^{s(1)} - \alpha_{x_1}^{s(x_1)} - \alpha_{x_2}^{s(x_2)} - \alpha_{1/2}^{s(1/2)} - \alpha_{3/2}^{s(3/2)} - \alpha_0^{s(0)}.$$

According to the algorithm, d must be a nonnegative integer. Further, we analyze all possible tuples of signs s and the corresponding values α. It is easy to verify that the unique tuple for which d is a nonnegative integer is

$$\alpha = \big(\alpha_\infty^+,\ \alpha_1^-,\ \alpha_{x_1}^-,\ \alpha_{x_2}^-,\ \alpha_{1/2}^-,\ \alpha_{3/2}^-,\ \alpha_0^-\big) = \left(\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ -\frac{1}{2},\ -\frac{1}{2},\ 1\right),$$

and $d = 0$. The corresponding function $\theta = \theta(x)$ calculated by (2.20) has the form

$$\theta = \frac{1}{x} + \frac{1}{4(x-1)} - \frac{1}{2x-1} - \frac{1}{2x-3} + \frac{1}{4(x-x_1)} + \frac{1}{4(x-x_2)}.$$

Step 3. For the set of values α obtained on Step 2, we search for a polynomial P of degree $d = 0$, which is a solution of the differential equation (2.21). Since the polynomial P has a zero degree, we substitute $P \equiv 1$ to Eq. (2.21). As a result, Eq. (2.21) takes the form

$$\frac{(4x_1x_2 - x_1 - x_2)(2x-3)^2}{16x(x-1)(2x-1)(x-x_1)(x-x_2)} = 0.$$

Therefore, $4x_1x_2 - x_1 - x_2 = 0$. For the initial parameters, this condition takes the form

$$\frac{4A_3B}{A_1A_3 + 4B(A_1 - A_3)} = 0.$$

Obviously, this condition does not hold for any physically admissible values of parameters. Thus, Eq. (6.5) has no Liouville solutions of the form (2.4).

Now we search for a solution of the form (2.9) for Eq. (6.5), i.e., a solution described in Case 2 of Theorem 1. The necessary conditions for the existence of such a solution are fulfilled (see Theorem 4). Now we apply the Kovacic algorithm as was described in Sec. 2.3.3.

Step 1. Let us define the following sets of integers:

$$E_1 = \{1,2,3\}, \qquad E_{x_1} = \{1,2,3\}, \qquad E_{x_2} = \{1,2,3\},$$
$$E_{1/2} = \{-2,2,6\}, \qquad E_{3/2} = \{-2,2,6\}, \qquad E_0 = \{4\}, \quad E_\infty = \{1,2,3\}.$$

Step 2. Consider all possible sets

$$s = \big(e_\infty,\ e_1,\ e_{x_1},\ e_{x_2},\ e_{1/2},\ e_{3/2},\ e_0\big)$$

of elements from E_∞, E_1, E_{x_1}, E_{x_2}, $E_{1/2}$, $E_{3/2}$, and E_0; at least one element in each set must be odd. Using (2.28), for each set s we obtain

$$d = \frac{1}{2}\big(e_\infty - e_1 - e_{x_1} - e_{x_2} - e_{1/2} - e_{3/2} - e_0\big).$$

According to the algorithm, d must be a nonnegative integer. Analyzing all possible sets s, we conclude that the unique set with nonnegative d is

$$e = \big(e_\infty,\ e_1,\ e_{x_1},\ e_{x_2},\ e_{1/2},\ e_{3/2},\ e_0\big) = \big(3,\ 1,\ 1,\ 1,\ -2,\ -2,\ 4\big),$$

and $d = 0$.

Step 3. Using (2.29), we construct the rational function θ for the chosen set e obtained on Step 2. We get

$$\theta = \frac{2}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x-x_1)} + \frac{1}{2(x-x_2)} - \frac{1}{x-1/2} - \frac{1}{x-3/2}.$$

A polynomial of degree $d = 0$ ($P \equiv 1$) should satisfy Eq. (2.30). We substitute $P \equiv 1$ to this equation and obtain

$$-\frac{3(4x_1x_2 - x_1 - x_2)(2x-3)^2}{8x^2(2x-1)^2(x-1)(x-x_1)(x-x_2)} = 0.$$

Hence, as in the previous case, we get

$$4x_1x_2 - x_1 - x_2 = 0.$$

Thus, Eq. (6.5) has no Liouville solutions of the form (2.9) for all physically admissible values of the parameters of the problem.

Now we search for a solution of the form (2.13) for Eq. (6.5), i.e., a solution described in Case 3 of Theorem 1. First, let us verify the necessary conditions for its existence (see Theorem 4). The function $S(x)$ has no poles of order greater than 2. The order of the pole at $x = \infty$ is greater than 1. The partial fraction expansion of $S(x)$ is (6.6). It can be easily shown that the remaining conditions of Theorem 4 hold:

$$\sqrt{1+4\alpha_i} = \frac{1}{2} \in \mathbb{Q} \quad (i = 1,4,5), \qquad \sqrt{1+4\alpha_j} = 2 \in \mathbb{Q} \quad (j = 2,3),$$

$$\sum_{i=0}^{5} \beta_i = 0, \quad \sqrt{1+4\gamma} = \frac{1}{2} \in \mathbb{Q}, \quad \gamma = -\frac{3}{16}.$$

Now we apply the Kovacic algorithm as described in Sec. 2.3.5.

Step 1. Let us define the following sets of integers:

$$E_\infty = \{3,4,5,6,7,8,9\}, \qquad E_1 = \{3,4,5,6,7,8,9\},$$
$$E_{x_1} = \{3,4,5,6,7,8,9\}, \qquad E_{x_2} = \{3,4,5,6,7,8,9\},$$
$$E_{1/2} = \{-6,-4,-2,0,2,4,6,8,10,12,14,16,18\},$$
$$E_{3/2} = \{-6,-4,-2,0,2,4,6,8,10,12,14,16,18\},$$
$$E_0 = \{12\}.$$

Step 2. Consider all possible sets

$$s = \big(e_\infty,\ e_1,\ e_{x_1},\ e_{x_2},\ e_{1/2},\ e_{3/2},\ e_0\big)$$

of elements from E_∞, E_1, E_{x_1}, E_{x_2}, $E_{1/2}$, $E_{3/2}$, and E_0. By the formula (2.41), we calculate d:

$$d = e_\infty - e_1 - e_{x_1} - e_{x_2} - e_{1/2} - e_{3/2} - e_0;$$

d must be nonnegative integer. Analyzing all possible sets of elements from E_∞, E_1, E_{x_1}, E_{x_2}, $E_{1/2}$, $E_{3/2}$, and E_0, we conclude that the unique set with a nonnegative d is

$$e = \bigl(e_\infty,\ e_1,\ e_{x_1},\ e_{x_2},\ e_{1/2},\ e_{3/2},\ e_0\bigr) = \bigl(9,\ 3,\ 3,\ 3,\ -6,\ -6,\ 12\bigr),$$

and $d = 0$.

Step 3. By the formula (2.42), we construct the function θ, using the set e obtained on Step 2. Then we get

$$\theta = \frac{12}{x} + \frac{3}{x - x_1} + \frac{3}{x - x_2} - \frac{6}{x - 1/2} - \frac{6}{x - 3/2} + \frac{3}{x - 1}.$$

Using (2.43), we construct the polynomial

$$W = x(x-1)(x-x_1)(x-x_2)(x-1/2)(x-3/2).$$

Further, the recursive formulas (2.44) are required:

$$P_{12} = -P, \quad P_{i-1} = -WP_i' + ((12-i)W' - W\theta)P_i - (12-i)(i+1)W^2 S(x) P_{i+1}, \quad P_{-1} = 0,$$

where $P_{12} = -P \equiv -1$ is a polynomial of degree $d = 0$. According to the algorithm, P_{-1} is identically zero; therefore, all its coefficients must be equal to zero. From this condition one can derive, using a computer algebra system, that $4x_1x_2 - 3x_1 - 3x_2 + 2 = 0$. It is easy to check that this condition is invalid for all physically admissible values of parameters.

Thus, we have proved that Eq. (6.5) with the coefficient $S(x)$ defined by (6.6) has no Liouville solutions for all physically admissible values of parameters of the problem. The theorem is proved. □

6.3. Special case $A_1A_3 + 4B(A_1 - A_3) = 0$. Now we assume that the parameters of the problem satisfy the condition (6.7). Then Eq. (6.3) has the form

$$\frac{d^2r}{dx^2} + d_1(x)\frac{dr}{dx} + d_2(x)r = 0, \tag{6.10}$$

where

$$d_1(x) = \frac{18 - 53x + 48x^2 - 12x^3}{2x(1-x)(1-2x)(3-2x)} - \frac{3(x - 2x_0)}{2x(x - x_0)},$$

$$d_2(x) = -\frac{(4x_0 - 3)(1-2x)^2}{16x(1-x)(3-2x)(x-x_0)}, \quad x_0 = \frac{A_1}{4(A_1 - A_3)} = -\frac{m\lambda^2}{A_3}.$$

After the change of variables (2.2), Eq. (6.10) can be rewritten as follows:

$$\frac{d^2y}{dx^2} = S_1(x)y, \tag{6.11}$$

where

$$S_1(x) = \frac{\beta_0}{x} + \frac{\beta_1}{x-1} + \frac{\alpha_1}{(x-1)^2} + \frac{\beta_2}{x-1/2} + \frac{\alpha_2}{(x-1/2)^2} + \frac{\beta_3}{x-3/2} + \frac{\alpha_3}{(x-3/2)^2} + \frac{\beta_4}{x-x_0} + \frac{\alpha_4}{(x-x_0)^2},$$

$$\alpha_1 = \alpha_4 = -\frac{3}{16}, \quad \alpha_2 = \alpha_3 = \frac{3}{4},$$

$$\beta_0 = -\frac{4x_0 - 3}{48x_0}, \quad \beta_1 = \frac{4x_0 - 9}{16(x_0 - 1)}, \quad \beta_2 = \frac{3}{2(2x_0 - 1)}, \quad \beta_3 = -\frac{8x_0 - 15}{6(2x_0 - 3)},$$

$$\beta_4 = \frac{32x_0^4 - 144x_0^3 + 204x_0^2 - 100x_0 + 3}{16x_0(x_0-1)(2x_0-1)(2x_0-3)}, \quad x_0 = \frac{A_1}{4(A_1 - A_3)}.$$

The Laurent expansion of $S_1(x)$ in a neighborhood of $x = \infty$ is

$$S_1(x)\big|_{x=\infty} \approx \frac{4x_0 - 3}{8x^2} + O(\frac{1}{x^3}).$$

Note that the explicit expression for x_0 implies that $x_0 < 0$. Thus, all poles of the function $S_1(x)$ are distinct, $\beta_0 \neq 0$, and the Laurent series of $S_1(x)$ at $x = \infty$ has order no greater than 2. Direct application of the Kovacic algorithm to the differential equation (6.11) yields the following result.

Theorem 15. *The differential equation* (6.11) *has no Liouville solutions for all physically admissible values of parameters.*

Proof. First, we search for a solution of the form (2.4) of Eq. (6.11), i.e., a solution described in Case 1 of Theorem 1. Note that the function $S_1(x)$ has four second-order finite poles, one first-order pole, and a second-order pole at $x = \infty$. Therefore, all conditions of Theorem 4 are fulfilled. Now we apply the Kovacic algorithm as was described in Sec. 2.3.1.

Step 1. Calculate the following values:

$$[\sqrt{S_1}]_1 = 0, \qquad \alpha_1^+ = \frac{3}{4}, \qquad \alpha_1^- = \frac{1}{4}, \qquad [\sqrt{S_1}]_{3/2} = 0, \qquad \alpha_{3/2}^+ = \frac{3}{2}, \qquad \alpha_{3/2}^- = -\frac{1}{2},$$

$$[\sqrt{S_1}]_{x_0} = 0, \qquad \alpha_{x_0}^+ = \frac{3}{4}, \qquad \alpha_{x_0}^- = \frac{1}{4}, \qquad [\sqrt{S_1}]_0 = 0, \qquad \alpha_0^+ = 1, \qquad \alpha_0^- = 1,$$

$$[\sqrt{S_1}]_{1/2} = 0, \qquad \alpha_{1/2}^+ = \frac{3}{2}, \qquad \alpha_{1/2}^- = -\frac{1}{2}, \qquad [\sqrt{S_1}]_\infty = 0, \qquad \alpha_\infty^\pm = \frac{1 \pm \sqrt{2x_0 - 0,5}}{2}.$$

Step 2. Since $x_0 < 0$, we have $2x_0 - 0.5 < 0$ and hence $\alpha_\infty^\pm$ are complex numbers. Therefore, d calculated by (2.19) is also a complex number and it cannot be a nonnegative integer, as the algorithm requires. Therefore, Eq. (6.11) has no Liouville solutions of the form (2.4).

Now we search for a solution of the form (2.9) for Eq. (6.11), i.e., a solution described in Case 2 of Theorem 1. The necessary conditions of the existence of such a solution to Eq. (6.11) are satisfied (see Theorem 4). Now we apply the Kovacic algorithm as was described in Sec. 2.3.3.

Step 1. Let us define the following sets of integers:

$$E_1 = \{1, 2, 3\}, \quad E_{x_0} = \{1, 2, 3\}, \quad E_{1/2} = \{-2, 2, 6\}, \quad E_{3/2} = \{-2, 2, 6\}, \quad E_0 = \{4\}, \quad E_\infty = \{2\}.$$

Step 2. Consider all possible sets

$$s = \bigl(e_\infty,\ e_1,\ e_{x_0},\ e_{1/2},\ e_{3/2},\ e_0\bigr)$$

of elements from E_∞, E_1, E_{x_0}, $E_{1/2}$, $E_{3/2}$, and E_0 with at least one odd element in each set. We calculate d for each set s by (2.28):

$$d = \frac{1}{2}\bigl(e_\infty - e_1 - e_{x_0} - e_{1/2} - e_{3/2} - e_0\bigr).$$

According to the algorithm, d must be nonnegative integer. Analyzing all possible sets of elements taken from E_∞, E_1, E_{x_0}, $E_{1/2}$, $E_{3/2}$, and E_0, we see that the unique set with nonnegative d is

$$e = \bigl(e_\infty,\ e_1,\ e_{x_0},\ e_{1/2},\ e_{3/2},\ e_0\bigr) = \bigl(2,\ 1,\ 1,\ -2,\ -2,\ 4\bigr),$$

and $d = 0$.

Step 3. By (2.29), we construct the function θ using the set e obtained on Step 2. Then we have

$$\theta = \frac{2}{x} + \frac{1}{2(x - x_0)} - \frac{2}{2x - 1} - \frac{2}{2x - 3} + \frac{1}{2(x - 1)}.$$

A polynomial of degree $d = 0$ ($P \equiv 1$) must satisfy Eq. (2.30). Substituting $P \equiv 1$ into (2.30) we obtain

$$\frac{3(4x_0 - 1)(2x - 3)^2}{8x^2(2x - 1)^2(x - 1)(x - x_0)} = 0.$$

Thus, we get $4x_0 - 1 = 0$, which contradicts the inequality $x_0 < 0$. Therefore, Eq. (6.11) has no solutions of the form (2.9).

Now we search for a solution of the form (2.13) for Eq. (6.11), i.e., a solution described in Case 3 of Theorem 1. Direct calculations show that not all necessary conditions (see Theorem 4) are satisfied. In particular, the value

$$\sqrt{1+4\gamma} = \sqrt{2x_0 - 0.5}$$

is pure imaginary since $x_0 < 0$. Consequently, this value is not real and rational, as the algorithm requires. This means that Eq. (6.11) has no Liouville solutions of the form (2.13). Finally, Eq. (6.11) has no Liouville solutions for all physically admissible parameters of the problem. The theorem is proved. □

6.4. Special case $B = \dfrac{9A_1A_3}{4(3A_3 - 4A_1)}$.

Assume that

$$A_1A_3 + 4B(A_1 - A_3) \neq 0$$

and the equation

$$\big(A_1A_3 + 4B(A_1 - A_3)\big)x^2 - 4B(A_1 - A_3)x + A_1B = 0$$

has the root $x = 3/2$. This means that the poles x_1 and $x = 3/2$ coincide (see (6.4)). We substitute $x = 3/2$ into the last equation and express the parameter B via other parameters. Then we get

$$B = \frac{9A_1A_3}{4(3A_3 - 4A_1)}. \tag{6.12}$$

Equation (6.12) has a physical sense if the inequalities $3A_3 > 4A_1$ and $2A_1 \geq A_3$ hold. In the case considered, Eq. (6.3) takes the following form:

$$\frac{d^2r}{dx^2} + d_1(x)\frac{dr}{dx} + d_2(x)r = 0, \tag{6.13}$$

where

$$\begin{aligned} d_1(x) &= \frac{18 - 53x + 48x^2 - 12x^3}{2x(1-x)(1-2x)(3-2x)} - \frac{27(A_1 - 2(A_1 - A_3)x)}{x\Delta}, \\ d_2(x) &= \frac{9(3A_3 - 2A_1)(1-2x)^2}{4x(1-x)(3-2x)\Delta}, \\ \Delta &= 4(5A_1 - 6A_3)x^2 - 36(A_1 - A_3)x + 9A_1. \end{aligned}$$

After the change of variables (2.2), Eq. (6.13) can be written as follows:

$$\frac{d^2y}{dx^2} = S_2(x)y, \tag{6.14}$$

where

$$\begin{aligned} S_2(x) = \frac{\beta_0}{x} + \frac{\beta_1}{x-1} + \frac{\alpha_1}{(x-1)^2} + \frac{\beta_2}{x-1/2} + \frac{\alpha_2}{(x-1/2)^2} + \frac{\beta_3}{x-3/2} \\ + \frac{\alpha_3}{(x-3/2)^2} + \frac{\beta_4}{x-x_0} + \frac{\alpha_4}{(x-x_0)^2}, \end{aligned} \tag{6.15}$$

$$\alpha_1 = \alpha_4 = -\frac{3}{16}, \quad \alpha_2 = \frac{3}{4}, \quad \alpha_3 = \frac{5}{16},$$

$$\beta_0 = -\frac{2x_0 - 3}{48x_0}, \quad \beta_1 = -\frac{3(2x_0+1)}{16(x_0-1)}, \quad \beta_2 = \frac{3(2x_0+1)}{4(2x_0-1)}, \quad \beta_3 = -\frac{8x_0 - 15}{12(2x_0-3)},$$

$$\beta_4 = -\frac{3(8x_0^3 - 24x_0^2 + 20x_0 - 1)}{16x_0(x_0-1)(2x_0-1)(2x_0-3)}, \quad x_0 = \frac{3A_1}{2(5A_1 - 6A_3)}.$$

The Laurent expansion of $S_2(x)$ in a neighborhood of $x = \infty$ is

$$S_2(x)\big|_{x=\infty} \approx -\frac{3}{16x^2} + O\left(\frac{1}{x^3}\right).$$

Applying the Kovacic algorithm to Eq. (6.14), we arrive at the following result.

Theorem 16. *The differential equation* (6.14) *has no Liouville solutions for all physically admissible values of parameters.*

Proof. First, we search for a solution of Eq. (6.14) of the form (2.4), i.e., a solution described in Case 1 of Theorem 1. Note that the function $S_2(x)$ has four finite second-order poles, one first-order pole, and a second-order pole at $x = \infty$. Consequently, all conditions of Theorem 4 hold. Now we start to apply the Kovacic algorithm step by step to search for the solution of form (2.4) for the differential equation (6.14) as described in Sec. 2.3.1.

Step 1. Let us calculate the following values:

$$[\sqrt{S_2}]_1 = 0, \quad \alpha_1^+ = \frac{3}{4}, \quad \alpha_1^- = \frac{1}{4}, \qquad [\sqrt{S_2}]_{3/2} = 0, \quad \alpha_{3/2}^+ = \frac{5}{4}, \quad \alpha_{3/2}^- = -\frac{1}{4},$$

$$[\sqrt{S_2}]_{x_0} = 0, \quad \alpha_{x_0}^+ = \frac{3}{4}, \quad \alpha_{x_0}^- = \frac{1}{4}, \qquad [\sqrt{S_2}]_0 = 0, \quad \alpha_0^+ = 1, \quad \alpha_0^- = 1,$$

$$[\sqrt{S_2}]_{1/2} = 0, \quad \alpha_{1/2}^+ = \frac{3}{2}, \quad \alpha_{1/2}^- = -\frac{1}{2}, \qquad [\sqrt{S_2}]_\infty = 0, \quad \alpha_\infty^+ = \frac{3}{4}, \quad \alpha_\infty^- = \frac{1}{4}.$$

Step 2. Since number ρ of the finite poles of the function $S_2(x)$ is equal to 5, then we have $2^{\rho+1} = 2^6 = 64$ tuples of signs

$$s = \big(s(\infty),\ s(1),\ s(x_0),\ s(1/2),\ s(3/2),\ s(0)\big).$$

For each tuple we calculate d by the formula (2.19):

$$d = \alpha_\infty^{s(\infty)} - \alpha_1^{s(1)} - \alpha_{x_0}^{s(x_0)} - \alpha_{1/2}^{s(1/2)} - \alpha_{3/2}^{s(3/2)} - \alpha_0^{s(0)}.$$

According to the algorithm, d must be a nonnegative integer. Further, we analyze all possible tuples of signs s and the corresponding values α. It is easy to verify that the unique tuple such that d is a nonnegative integer is

$$\alpha = \big(\alpha_\infty^+,\ \alpha_1^-,\ \alpha_{x_0}^-,\ \alpha_{1/2}^-,\ \alpha_{3/2}^-,\ \alpha_0^-\big) = \left(\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ -\frac{1}{2},\ -\frac{1}{4},\ 1\right),$$

and $d = 0$. The function $\theta = \theta(x)$, defined by (2.20), for the chosen set of values α has the form

$$\theta = \frac{1}{x} + \frac{1}{4(x-1)} - \frac{1}{2x-1} - \frac{1}{4x-6} + \frac{1}{4(x-x_0)}.$$

Step 3. For the set of values α obtained on the previous step, we search for a polynomial P of degree $d = 0$ that satisfy the differential equation (2.21). Substituting $P \equiv 1$ into (2.21) we get

$$\frac{(10x_0 - 3)(2x - 3)}{16x(x-1)(2x-1)(x-x_0)} = 0.$$

Thus, $10x_0 - 3 = 0$. Using the explicit expression for x_0, we obtain $A_3 = 0$. This contradicts the assumption that the moments of inertia are positive. Hence Eq. (6.14) has no Liouville solutions of the form (2.4).

Now we search for a solution of the form (2.9) for Eq. (6.14), i.e., a solution described in Case 2 of Theorem 1. The necessary conditions of the existence are fulfilled. Now we apply the Kovacic algorithm (see Sec. 2.3.3).

Step 1. Let us define the following sets of integers:

$$E_1 = \{1,2,3\}, \qquad E_{x_0} = \{1,2,3\}, \qquad E_{1/2} = \{-2,2,6\},$$
$$E_{3/2} = \{-1,2,5\}, \qquad E_0 = \{4\}, \qquad E_\infty = \{1,2,3\}.$$

Step 2. Consider all possible sets

$$s = \big(e_\infty,\ e_1,\ e_{x_0},\ e_{1/2},\ e_{3/2},\ e_0\big)$$

of elements from E_∞, E_1, E_{x_0}, $E_{1/2}$, $E_{3/2}$, and E_0 with at least one odd element in each set. We calculate d for each set s by the formula (2.28):

$$d = \frac{1}{2}\big(e_\infty - e_1 - e_{x_0} - e_{1/2} - e_{3/2} - e_0\big).$$

According to the algorithm, d must be nonnegative integer. By analyzing all possible sets of elements taken from E_1, E_{x_0}, $E_{1/2}$, $E_{3/2}$, E_0, and E_∞, we conclude that the unique set with nonnegative d is

$$e = \big(e_\infty,\ e_1,\ e_{x_0},\ e_{1/2},\ e_{3/2},\ e_0\big) = \big(3,\ 1,\ 1,\ -2,\ -1,\ 4\big),$$

and $d = 0$.

Step 3. Using the set e obtained on Step 2, we construct the function θ by the formula (2.29):

$$\theta = \frac{2}{x} + \frac{1}{2(x-x_0)} - \frac{2}{2x-1} - \frac{1}{2x-3} + \frac{1}{2(x-1)}.$$

A polynomial of degree $d = 0$ ($P \equiv 1$) must satisfy Eq. (2.30). Substituting $P \equiv 1$ into (2.30) we obtain

$$-\frac{3(10x_0-3)(2x-3)}{8x^2(2x-1)^2(x-1)(x-x_0)} = 0.$$

Hence $10x_0 - 3 = 0$. This condition was already considered above: we proved that it is invalid for all physically admissible values of parameters of the problem. Thus, Eq. (6.14) has no Liouville solutions of the form (2.9).

Now we search for a solution of the form (2.13) for Eq. (6.14), i.e., a solution described in Case 3 of Theorem 1. First, we verify the necessary conditions of the existence of such a solution (see Theorem 4). The function $S_2(x)$ has no poles of order greater than 2. The order of a pole of $S_2(x)$ at $x = \infty$ is greater than 1. The partial fraction expansion of $S_2(x)$ has the form (6.15). Direct calculations show that all other conditions of Theorem 4 are satisfied:

$$\sqrt{1+4\alpha_i} = \frac{1}{2} \in \mathbb{Q} \quad (i = 1,4), \qquad \sqrt{1+4\alpha_2} = 2 \in \mathbb{Q}, \qquad \sqrt{1+4\alpha_3} = \frac{3}{2} \in \mathbb{Q},$$
$$\sum_{i=0}^{4} \beta_i = 0, \qquad \sqrt{1+4\gamma} = \frac{1}{2} \in \mathbb{Q}, \qquad \gamma = -\frac{3}{16}.$$

Now we apply the Kovacic algorithm (see Sec. 2.3.5).

Step 1. Let us define the following sets of integers:

$$E_1 = \{3,4,5,6,7,8,9\}, \qquad E_{x_0} = \{3,4,5,6,7,8,9\},$$
$$E_{1/2} = \{-6,-4,-2,0,2,4,6,8,10,12,14,16,18\},$$
$$E_{3/2} = \{-3,0,3,6,9,12,15\}, \qquad E_0 = \{12\}, \qquad E_\infty = \{3,4,5,6,7,8,9\}.$$

Step 2. Consider all possible sets

$$s = \big(e_\infty,\ e_1,\ e_{x_0},\ e_{1/2},\ e_{3/2},\ e_0\big)$$

of elements from E_∞, E_1, E_{x_0}, $E_{1/2}$, $E_{3/2}$, and E_0 and calculate d by the formula (2.41):

$$d = e_\infty - e_1 - e_{x_0} - e_{1/2} - e_{3/2} - e_0.$$

Analyzing all possible sets of elements taken from E_∞, E_1, E_{x_0}, $E_{1/2}$, $E_{3/2}$, and E_0, we conclude that the unique set with nonnegative integer d is

$$e = \big(e_\infty,\ e_1,\ e_{x_0},\ e_{1/2},\ e_{3/2},\ e_0\big) = \big(9,\ 3,\ 3,\ -6,\ -3,\ 12\big),$$

and $d = 0$.

Step 3. By (2.42), we construct the function θ using the set e obtained on the previous step. Then we get

$$\theta = \frac{12}{x} + \frac{3}{x - x_0} - \frac{6}{x - 1/2} - \frac{3}{x - 3/2} + \frac{3}{x - 1}.$$

Using (2.43), we construct the polynomial

$$W = x(x-1)(x-x_0)(x-1/2)(x-3/2).$$

Further, we need the recursive formulas (2.44):

$$P_{12} = -P, \quad P_{i-1} = -WP_i' + ((12-i)W' - W\theta)P_i - (12-i)(i+1)W^2 S_2(x) P_{i+1}, \quad P_{-1} = 0,$$

where $P_{12} = -P \equiv -1$ is a polynomial of degree $d = 0$. According to the algorithm, P_{-1} is an identically zero polynomial; therefore, all its coefficients are equal to zero. From this condition, using a computer algebra system, we can obtain that $10x_0 - 3 = 0$. It was proved above that this condition does not hold for all physically admissible values of parameters.

Thus, we have verified that Eq. (6.14) has no Liouville solutions. The theorem is proved. ☐

6.5. Special case $B = \dfrac{A_1^2}{4\,(A_3 - A_1)}$.

Assume that

$$A_1A_3 + 4B(A_1 - A_3) \neq 0, \quad x_1 = x_2.$$

From (6.4) we conclude that the poles coincide if

$$B = \frac{A_1^2}{4(A_3 - A_1)}. \tag{6.16}$$

The condition (6.16) is physically admissible if $A_3 > A_1$. In this case, Eq. (6.3) has the following form:

$$\frac{d^2r}{dx^2} + d_1(x)\frac{dr}{dx} + d_2(x)r = 0, \tag{6.17}$$

where

$$d_1(x) = \frac{18 - 53x + 48x^2 - 12x^3}{2x(1-x)(1-2x)(3-2x)} + \frac{3x_0}{x(x-x_0)},$$

$$d_2(x) = \frac{x_0(2x_0-3)(1-2x)^2}{8x(1-x)(3-2x)(x-x_0)^2}, \quad x_0 = \frac{A_1}{2(A_1 - A_3)} = -\frac{2m\lambda^2}{A_1}.$$

After the change of variables (2.2), the differential equation (6.17) takes the form

$$\frac{d^2y}{dx^2} = S_3(x)y, \tag{6.18}$$

where

$$S_3(x) = \frac{\beta_0}{x} + \frac{\beta_1}{x-1} + \frac{\alpha_1}{(x-1)^2} + \frac{\beta_2}{x-1/2} + \frac{\alpha_2}{(x-1/2)^2} + \frac{\beta_3}{x-3/2} + \frac{\alpha_3}{(x-3/2)^2} + \frac{\beta_4}{x-x_0} + \frac{\alpha_4}{(x-x_0)^2}, \tag{6.19}$$

$$\alpha_1 = -\frac{3}{16}, \quad \alpha_2 = \alpha_3 = \frac{3}{4}, \quad \alpha_4 = -\frac{4x_0^2 - 10x_0 + 7}{8(x_0-1)},$$

$$\beta_0 = -\frac{2x_0 - 3}{24x_0}, \quad \beta_1 = \frac{2x_0^2 - 9x_0 + 6}{8(x_0-1)^2}, \quad \beta_2 = \frac{3}{2x_0 - 1}, \quad \beta_3 = -\frac{4x_0 - 9}{3(2x_0-3)},$$

$$\beta_4 = \frac{16x_0^5 - 112x_0^4 + 260x_0^3 - 256x_0^2 + 94x_0 - 3}{8x_0(2x_0-1)(2x_0-3)(x_0-1)^2}, \quad x_0 = \frac{A_1}{2(A_1 - A_3)}.$$

The Laurent expansion of $S_3(x)$ at $x = \infty$ has the form

$$S_3(x)\big|_{x=\infty} \approx -\frac{3}{16x^2} + O\left(\frac{1}{x^3}\right).$$

This special case has the following peculiarity: the coefficient α_4 in the partial fraction expansion of the function $S_3(x)$ depends on the parameters. This coefficient has no definite numerical value but it is determined by the expression

$$\alpha_4 = -\frac{4x_0^2 - 10x_0 + 7}{8(x_0-1)}.$$

As a result, the value d can be arbitrarily large. Recall that d is a degree of the polynomial P calculated in every case of the algorithm. Therefore, we consider only the case where $d = 0$. Direct application of the Kovacic algorithm to the differential equation (6.18) yields the following result.

Theorem 17. *Assume that $d = 0$ and the condition* (6.9) *holds. Then the differential equation* (6.18) *has no Liouville solutions for all physically admissible values of parameters.*

Proof. First, we search for a solution of Eq. (6.18) of the form (2.4), i.e., a solution described in Case 1 of Theorem 1. Note that the function $S_3(x)$ has four second-order finite poles, one first-order pole, and a second-order pole at $x = \infty$. Therefore, the conditions of Theorem 4 necessary for the existence of a solution of the form (2.4) for the differential equation (6.18) are satisfied. Now we apply the Kovacic algorithm (see Sec. 2.3.1).

Step 1. We introduce the notation $b_0 = 1 + 4\alpha_4$ and calculate the following values:

$$\begin{array}{lll}
[\sqrt{S_3}]_1 = 0, & \alpha_1^+ = \frac{3}{4}, & \alpha_1^- = \frac{1}{4}, \\
[\sqrt{S_3}]_{1/2} = 0, & \alpha_{1/2}^+ = \frac{3}{2}, & \alpha_{1/2}^- = -\frac{1}{2}, \\
[\sqrt{S_3}]_{3/2} = 0, & \alpha_{3/2}^+ = \frac{3}{2}, & \alpha_{3/2}^- = -\frac{1}{2}, \\
[\sqrt{S_3}]_0 = 0, & \alpha_0^+ = 1, & \alpha_0^- = 1, \\
[\sqrt{S_3}]_{x_0} = 0, & \alpha_{x_0}^+ = \frac{1}{2} + \frac{1}{2}\sqrt{b_0}, & \alpha_{x_0}^- = \frac{1}{2} - \frac{1}{2}\sqrt{b_0}, \\
[\sqrt{S_3}]_\infty = 0, & \alpha_\infty^+ = \frac{3}{4}, & \alpha_\infty^- = \frac{1}{4}.
\end{array}$$

Step 2. Since the number ρ of finite poles of the function $S_3(x)$ is equal to 5, we have $2^{\rho+1} = 2^6 = 64$ tuples of signs

$$s = \big(s(\infty),\ s(1),\ s(1/2),\ s(3/2),\ s(0),\ s(x_0)\big).$$

Choose signs in the tuples s in such a way that the value

$$d = \alpha_\infty^{s(\infty)} - \alpha_1^{s(1)} - \alpha_{1/2}^{s(\frac{1}{2})} - \alpha_{3/2}^{s(\frac{3}{2})} - \alpha_0^{s(0)} - \alpha_{x_0}^{s(x_0)},$$

calculated by (2.19), is equal to zero for some b_0. Note that since $\alpha_0^+ = \alpha_0^- = 1$, the tuple with the sign $+$ chosen for $s(0)$ and the tuple with the sign $-$ chosen for $s(0)$ are equal if all remaining signs in these tuples are the same. The tuples of signs s and the corresponding values b_0 with $d = 0$ are listed below. In all tuples the sign $+$ is chosen for $s(0)$:

$$\begin{aligned}
&s_1 = (+,-,-,-,+,+), && b_0 = 0; && s_{10} = (+,+,+,-,+,-), && b_0 = 25;\\
&s_2 = (+,-,-,-,+,-), && b_0 = 0; && ,s_{11} = (+,+,-,+,+,-), && b_0 = 25;\\
&s_3 = (-,-,-,+,-,-), && b_0 = 1; && s_{12} = (-,+,+,-,+,-), && b_0 = 36;\\
&s_4 = (+,+,-,-,+,-), && b_0 = 1; && s_{13} = (-,+,-,+,+,-), && b_0 = 36;\\
&s_5 = (-,+,-,-,+,-), && b_0 = 4; && s_{14} = (+,-,+,+,+,-), && b_0 = 64;\\
&s_6 = (+,-,+,-,+,-), && b_0 = 16; && s_{15} = (-,-,+,+,+,-), && b_0 = 81;\\
&s_7 = (+,-,-,+,+,-), && b_0 = 16; && s_{16} = (+,+,+,+,+,-), && b_0 = 81;\\
&s_8 = (-,-,-,+,+,-), && b_0 = 25; && s_{17} = (-,+,+,+,+,-), && b_0 = 100.\\
&s_9 = (-,-,+,-,+,-), && b_0 = 25;
\end{aligned}$$

Similarly, one can list 17 other tuples with the sign $-$ chosen for $s(0)$. Let us consider in more detail the case where the tuple s_1 is chosen; the remaining cases can be examined similarly. Further, using the formula (2.20), we construct the function θ using the values $\alpha_c^\pm$ corresponding to the signs chosen for the tuple s_1. Then the function θ has the form

$$\theta = \frac{1}{4(x-1)} - \frac{1}{2x-1} - \frac{1}{2x-3} + \frac{1}{x} + \frac{1}{2(x-x_0)}.$$

Step 3. A polynomial of degree $d = 0$ ($P \equiv 1$) must satisfy Eq. (2.21). We substitute $P \equiv 1$ into (2.21) and get

$$\frac{x_0(2x_0-1)(2x-3)^2}{8x(x-1)(2x-1)(x-x_0)^2}.$$

Since $x_0 \neq 0$ and $2x_0 - 1 \neq 0$, this condition cannot be satisfied identically. Hence, the tuple s_1 does not give a solution of the form (2.4) for the differential equation (6.18). Similarly, one can consider all remaining tuples of signs and ascertain that Eq. (6.18) does not possess any Liouville solution of the form (2.4).

Now we search for a solution of the form (2.9) for the differential equation (6.18), i.e., a solution described in Case 2 of Theorem 1. The necessary conditions for the existence of such a solution hold (see Theorem 4). Now we apply the Kovacic algorithm (see Sec. 2.3.3).

Step 1. Let us define the following sets of integers:

$$E_1 = \{1,2,3\}, \quad E_{1/2} = \{-2,2,6\}, \quad E_{3/2} = \{-2,2,6\}, \quad E_0 = \{4\},$$
$$E_{x_0} = \left\{(2 + k\sqrt{b_0}) \cap \mathbb{Z},\ k = 0, \pm 2\right\}, \quad E_\infty = \{1,2,3\}.$$

Step 2. Consider all possible sets

$$s = \left(e_\infty,\ e_1,\ e_{1/2},\ e_{3/2},\ e_0,\ e_{x_0}\right)$$

of elements from E_∞, E_1, $E_{1/2}$, $E_{3/2}$, E_0, and E_{x_0} and in each set at least one element is odd. For each set s, we calculate d by (2.28):

$$d = \frac{1}{2}\big(e_\infty - e_1 - e_{1/2} - e_{3/2} - e_0 - e_{x_0}\big).$$

As above, we assume that $d = 0$. Execution of this step of the algorithm is concerned with a large number of possible sets s. Thus, we present here a detailed investigation for only one case; all remaining cases can be examined similarly. Choose the set

$$s_1 = \big(e_\infty,\ e_1,\ e_{1/2},\ e_{3/2},\ e_0,\ e_{x_0}\big) = \big(3,\ 1,\ -2,\ -2,\ 4,\ 2\big).$$

Step 3. By (2.29), we construct the function θ using elements of the set s_1. Hence, θ has the form

$$\theta = \frac{1}{2(x-1)} - \frac{2}{2x-1} - \frac{2}{2x-3} + \frac{2}{x} + \frac{1}{x-x_0}.$$

A polynomial of degree $d = 0$ ($P \equiv 1$) must satisfy Eq. (2.30). We substitute $P \equiv 1$ into (2.30) and obtain

$$-\frac{3x_0(2x_0-1)(2x-3)^2}{4x^2(x-x_0)^2(2x-1)^2(x-1)} = 0.$$

Hence, this condition is valid if $x_0 = 0$ or $2x_0 - 1 = 0$. As was shown above, none of these conditions hold.

Thus, we have proved that for the set s_1 of elements from E_∞, E_1, $E_{1/2}$, $E_{3/2}$, E_0, and E_{x_0}, the differential equation (6.18) has no Liouville solutions of the form (2.9). Note that a similar analysis was performed for all other sets s with $d = 0$. As a result, we proved the nonexistence of solutions of the form (2.9) for the differential equation (6.18) for any set s.

Finally, we search for a solution of the form (2.13) for the differential equation (6.18), i.e., a solution described in Case 3 of Theorem 1. First, check whether the necessary conditions for its existence hold (see Theorem 4). The function $S_3(x)$ has no poles of order greater than 2. The order of pole of $S_3(x)$ at $x = \infty$ is greater than 1. The partial fraction expansion of $S_3(x)$ is (6.19). The following conditions hold:

$$\sqrt{1+4\alpha_1} = \frac{1}{2} \in \mathbb{Q}, \quad \sqrt{1+4\alpha_i} = 2 \in \mathbb{Q} \quad (i = 2, 3),$$
$$\sum_{i=0}^{4} \beta_i = 0, \quad \sqrt{1+4\gamma} = \frac{1}{2} \in \mathbb{Q}, \quad \gamma = -\frac{3}{16}.$$

Assume that the condition

$$\sqrt{1+4\alpha_4} = \frac{3-2x_0}{\sqrt{2(1-x_0)}} \in \mathbb{Q},$$

is valid; otherwise, Eq. (6.18) obviously has no Liouville solutions of the form (2.13). Now we apply the Kovacic algorithm (see Sec. 2.3.5).

Step 1. Let us define the sets

$$E_1 = \{3,4,5,6,7,8,9\}, \quad E_0 = \{12\},$$
$$E_{1/2} = \{-6,-4,-2,0,2,4,6,8,10,12,14,16,18\},$$
$$E_{3/2} = \{-6,-4,-2,0,2,4,6,8,10,12,14,16,18\},$$
$$E_{x_0} = \big\{(6 + k\sqrt{b_0}) \cap \mathbb{Z},\ k = 0, \pm 1, \dots, \pm 6)\big\},$$
$$E_\infty = \{3,4,5,6,7,8,9\}.$$

Step 2. Consider all possible sets

$$s = \bigl(e_\infty,\ e_1,\ e_{1/2},\ e_{3/2},\ e_0,\ e_{x_0}\bigr)$$

of elements from E_∞, E_1, $E_{1/2}$, $E_{3/2}$, E_0, and E_{x_0} and calculate d by (2.41):

$$d = e_\infty - e_1 - e_{1/2} - e_{3/2} - e_0 - e_{x_0}.$$

Step 3. As above, we assume that $d = 0$. Among all sets s, we choose those with $d = 0$. Considering quantities of elements of sets E_∞, E_1, $E_{1/2}$, $E_{3/2}$, E_0, and E_{x_0} introduced on the first step, we can estimate that even if we fix one element of E_{x_0}, we must examine $7\cdot 7\cdot 13\cdot 13 = 8281$ sets s, for each of which $d = 0$. Therefore, we illustrate the analysis on a typical example instead of listing all possible cases. Choose the set s_1 with $d = 0$:

$$s_1 = \bigl(e_\infty,\ e_1,\ e_{1/2},\ e_{3/2},\ e_0,\ e_{x_0}\bigr) = \bigl(9,\ 3,\ 0,\ 0,\ 12,\ -6\bigr).$$

By the formula (2.42), we construct the function θ using the set s_1. The function θ has the form

$$\theta = \frac{3}{x-1} + \frac{12}{x} - \frac{6}{x-x_0}.$$

According to (2.43), we construct the polynomial

$$W = x(x-1)(x-\frac{1}{2})(x-\frac{3}{2})(x-x_0).$$

Further, the recursive formulas (2.44) are required:

$$P_{12} = -P, \quad P_{i-1} = -WP_i' + ((12-i)W' - W\theta)P_i - (12-i)(i+1)W^2 S_3(x)P_{i+1}, \quad P_{-1} = 0,$$

where $P_{12} = -P = -1$ is a polynomial of degree $d = 0$. According to the algorithm, P_{-1} is a polynomial that must be identically zero. Therefore, all its coefficients are equal to zero. These coefficients include one unknown variable x_0. The corresponding system of equations is inconsistent. All other sets s can be considered similarly. This means that Eq. (6.18) has no Liouville solutions of the form (2.13) in the case $d = 0$. The theorem is proved. □

A similar analysis was conducted for all sets with $d = 1, 2, 3, 4$. In all these cases, the differential equation (6.18) has no Liouville solutions. In summary, in the problem of the motion of a spindle-shaped body on a fixed, perfectly rough horizontal plane we have found a unique case where the second-order linear differential equation (6.2) has a Liouville solution. This is the case where Mushtari's condition (6.1) hold.

7. Conclusion

Application of the Kovacic algorithm to the problem of motion of a heavy, rotationally symmetric rigid body on a perfectly rough horizontal plane allows one to prove the nonexistence of Liouville solutions in the case where the moving body is an infinitely thin round disk or a disk of finite thickness. In the case where the moving body is a dynamically symmetric torus, the Kovacic algorithm allows one to prove the nonexistence of Liouville solutions for almost all values of parameters of the problem. On the contrary, if the moving body is a dynamically symmetric paraboloid, the corresponding second-order linear differential equation possesses Liouville solutions for all physically admissible values of parameters of the problem. The explicit form of these solutions is obtained. Using these Liouville solutions, we give a qualitative description of the motion of a paraboloid on a plane. The trajectory of the contact point M on the surface of the paraboloid is a curve consisting of periodically repeating waves and tangent to two parallels of the paraboloid. The trajectory of the contact point on the supporting plane has the same pattern and it lies between two concentric circles. During the motion of the paraboloid, the contact point M touches these two circles in turn. Steady motions of the paraboloid on a perfectly rough plane are found and their stability is investigated. In the problem of the motion of a spindle-shaped body on a perfectly rough horizontal plane, the Kovacic algorithm allows one to prove the nonexistence of Liouville solutions for almost all values of parameters of the problem, except for the case where these parameters satisfy the Mushtari condition (see [35]).

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