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Continuum Mechanics, Applied Mathematics and Scientific Computing: Godunov's Legacy

A Liber Amicorum to Professor Godunov

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S.K. Godunov and Kinetic Theory in KIAM RAS

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Abstract The article describes the history of the development of cooperation between scientists from the Keldysh Institute of Applied Mathematics of the Russian Academy of Sciences with S. K. Godunov. A lot of interesting results have been established in the theory of kinetic equations and computational mathematics in the process of this cooperation.

In 1971, a review [1] was published. The review was the first article in the USSR on discrete models of the Boltzmann equation, contained a study of a model with three speeds. As it turned out later, Boltzmann wrote a similar model in 1872, but now this model has become known as the Godunov–Sultangazin model, although abroad it is called the one-dimensional Broadwell model ($d = 1, n = 3$):

$$\begin{cases} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} = f_2^2 - f_1 f_3, \\ \frac{\partial f_2}{\partial t} = 2(f_1 f_3 - f_2^2), \\ \frac{\partial f_3}{\partial t} - \frac{\partial f_3}{\partial x} = f_2^2 - f_1 f_3. \end{cases} \tag{1}$$

Here is an example where this name is included in the title of [2]. The relationship between the four-velocity Broadwell model ($d = 2, n = 4$)

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$$\begin{cases} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} = f_3 f_4 - f_2 f_1, \\ \frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} = f_3 f_4 - f_2 f_1, \\ \frac{\partial f_3}{\partial t} + \frac{\partial f_3}{\partial y} = f_1 f_2 - f_3 f_4, \\ \frac{\partial f_4}{\partial t} - \frac{\partial f_4}{\partial y} = f_1 f_2 - f_3 f_4 \end{cases}$$

and this Godunov–Sultangazin is also discussed in the review. The review played a significant role in the development of both the theory of discrete models of the Boltzmann equation and for the development of the practice of discrete modeling. In 1971, in the Department of Kinetic Equations, M. V. Maslennikov organized a seminar on the Boltzmann equation, where, in particular, the book of the Swedish mathematician Torstein Carleman, “Mathematical Problems of the Kinetic Theory of Gases”, was analyzed. This book had an application where a model with two velocities

$$\begin{cases} \frac{\partial f_1}{\partial t} + v_1 \frac{\partial f_1}{\partial x} = f_2^2 - f_1^2, \\ \frac{\partial f_2}{\partial t} + v_2 \frac{\partial f_2}{\partial x} = f_2^2 - f_1^2 \end{cases}$$

was introduced, which is now called Carleman model ($d = 1$, $n = 2$). Someone remembered (maybe it was M. V. Maslennikov) that a review of Godunov and Sultangazin had just appeared. After that, they began to analyze this review. One of the authors (V. V.—later I) reported on both works (I was a regular reserve speaker). Therefore, it was necessary to read carefully, and in the review, many questions, tasks, and hints were formulated. One of these hints on the task looked like this. “At first glance, the model system chosen by us is not much more complicated than the Carleman model. However, a detailed analysis of this system shows that it does not have quadratic dissipating integrals, and therefore it is difficult to obtain an existence theorem for it covering all positive times $0 < t < \infty$.” I took this question to investigate: what is the difference between the two models, what are the dissipative integrals there, and what will be for the Boltzmann equation? The theorem on the uniqueness of the Boltzmann H-function was obtained: for a gas described by the Boltzmann equation, entropy is the only extensive (additive) functional that increases with time. It also clarified the reason for the uniqueness of the H-function as a dissipative integral for the model of both Broadwell and Godunov–Sultangazin and non-uniqueness for Carleman model. This was the main result of both of my theses and was also published in UMN [3].

Sergei Konstantinovich often appeared in the Keldysh Institute. In one of these visits, he drew attention to the double-divergent form of the hydrodynamic consequences of discrete models (1) of the Boltzmann equation:

$$\frac{\partial L_{q_i}}{\partial t} + \frac{\partial M_{q_i}}{\partial x} = 0,$$

$$L = \exp(q_1 + q_2 - 1) + \exp(q_1 - 1) + \exp(q_1 - q_2 - 1),$$

$$M = \exp(q_1 + q_2 - 1) - \exp(q_1 - q_2 - 1),$$

$$f_1 = \exp(q_1 + q_2 - 1), \quad f_2 = \exp(q_2 - 1), \quad f_3 = \exp(-q_1 - q_2 - 1)$$

$$(q_1, q_2 \text{ are new variables}).$$

I did not react, and in the next visit, he presented an extract from his review for a simple discrete model on one page. After that, we got down to research: this form was obtained for the Boltzmann equation (which was done, as it turned out, by the review authors themselves) and for its quantum analogs:

$$\frac{\partial L_{\alpha_\mu}}{\partial t} + \operatorname{div} \mathbf{M}_{\alpha_i} = 0, \quad i = 0, \ldots, 4,$$

$$L(\alpha) = -\frac{1}{\theta} \int \ln \left(1 - \theta \exp\left(\sum_i \varphi_i(\mathbf{v})\alpha_i\right)\right) dv,$$

where

$$\varphi_0 = 1, \ \ \varphi_1 = v_1, \ \ \varphi_2 = v_2, \ \ \varphi_3 = v_3, \ \ \varphi_4 = \mathbf{v}^2,$$

and parameters α_i are connected with usual macroscopic parameters density ρ, temperature T, and mean velocity $\mathbf{u}$ by formulae

$$\alpha_4 = -\frac{1}{2kT}, \ \ \alpha_i = \frac{u_i}{kT}, \ \ \alpha_0 = \ln \rho - \frac{3}{2} \ln(2\pi kT) - \frac{\mathbf{u}^2}{2kT}.$$

This connection is obtained by comparison of two different forms of Maxwellian:

$$f_0(v) \equiv \frac{\rho}{(2\pi kT)^{3/2}} \exp\left(-\frac{(\mathbf{v}-\mathbf{u})^2}{2kT}\right) = \exp\left(\alpha_0 + \sum_i \alpha_i v_i + \alpha_4 v^2\right) =$$

$$= \exp\left(\alpha_0 - \frac{\boldsymbol{\alpha}^2}{4\alpha_4}\right) \exp\left(\alpha_4 \left(v + \frac{\boldsymbol{\alpha}^2}{2\alpha_4}\right)\right), \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3),$$

where θ is the indicator of statistics: $\theta \to 0$ denotes the Boltzmann statistics, $\theta = -1$ denotes Bose statistics, and $\theta = +1$ denotes Fermi statistics.

This was published in the book [4] on kinetic equations, where these forms of Godunov were included. But S. K. Godunov did not stop: he proposed to do the same for the Vlasov equation. Here, too, an opportunity turned up: M. A. Negmatov and I began writing an article on the Vlasov equation. The reviewer wrote to us that the main result of the novelty is just the form of Godunov, and even proposed to change the name. We did just that. It turned out even two articles with the name of Godunov [5, 6].

Here is another comment from this review of Godunov and Sultangazin in 1971, which had consequences: "It should be noted that the method of choosing discrete velocities, which provide a description of gas flow with the conservation laws of three components of momentum and with energy conservation, has not been developed in the general case. This development is associated with difficulties of a combinatorial-geometric nature." Now we can firmly say: developed. A somewhat detective story that influenced the development of European education. Going from one foreign country to another, Alexander Vasilyevich Bobylev entered our room and said: "Here, Vitya, look". And he gave his joint article with Carlo Cercignani. The pathos of the article was in the construction of the first discrete models of the Boltzmann equation for mixtures. I looked at the models, and became suspicious, deciding to check the number of invariants. The fact is that a discrete model must contain exactly as many invariants as the original equation. In the Boltzmann equation, this is energy, the number of particles and components of an impulse, which means in a discrete model it should be the same, otherwise we have a catastrophe: if there is not enough or too many invariants, then a final distribution is not Maxwellian, and the hydrodynamic consequences are not correct. We already had a procedure for calculating invariants, and in the first of the models, there was one extra invariant. In the West, such invariants have the special name spurious invariants. The other model had twenty-five variables (the first twelve), and manually counting the number of invariants was difficult. But we knew that Alexander Dmitrievich Bruno had a special program for calculating Hamiltonian system invariants. In addition, we already knew that the algebra of conservation laws for discrete models of the Boltzmann equation and for chemical kinetics equations is the same as for Hamiltonian systems. I called A. D. Bruno, and he connected me with his graduate student A. Aranson, who did this. He found two extra invariants. I asked Viktor I. Turchaninov to check up, the result was the same. At that time we prepared for publishing an article just on the relation between conservation laws for discrete models of the Boltzmann equation and quantum Hamiltonians, and we included these results there [7]. When A. V. Bobylev returned, I showed the results, he immediately agreed, and even asked what kind of invariants these were. Again we turned to Aranson, he also presented the extra invariants themselves. A. V. Bobylev just at that time got a place in Sweden, in Karlstadt, where his task was to do University "from the Ryazan State Pedagogical College", as he said. This meant to bring up several candidates of science (Ph.D.). He involved his two post-graduate students, Miriella Vinerian and Nicholas Bernhof, in this topic, and things went forward [8, 9]. Miriella Vinerian and Nicholas Bernhof fulfilled Ph.D., got married, and named their first child Alexander (as Bobylev). So our M. V. Keldysh Institute of Applied Mathematics of RAS with the efforts of several departments made a comprehensive contribution to the development of the Swedish and all-European civilization. These topics, which started in this way from the review of Godunov and Sultangazin [1], went on to develop further: conservation laws for the equations of chemical kinetics and the Liouville equation [10–14]. And the results for the Vlasov equation are now transferred to the Vlasov–Maxwell–Einstein equations [15].

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