Saddle-Point Method in Terminal Control with Sections in Phase Constraints

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Abstract. A new approach to solving terminal control problems with phase constraints, based on saddle-point sufficient optimality conditions, is considered. The basis of the approach is Lagrangian formalism and duality theory. We study linear controlled dynamics in the presence of phase constraints. The cross section of phase constraints at certain points in time leads to the appearance of new intermediate finite-dimensional convex programming problems. In fact, the optimal control problem, defined over the entire time interval, is split into a number of independent intermediate subproblems, each of which is defined in its own subsegment. Combining the solutions of these subproblems together, we can obtain solutions5 to the original problem on the entire time interval. To this end, a gradient flow is launched to solve all intermediate problems at the same time. The convergence of computing technology to the solution of the optimal control problem in all variables is proved.

Keywords: Optimal control · Lagrange function · Duality · Saddle point · Iterative solution methods · Convergence

1 Introduction

Dynamic problems of terminal control under state constraints are among the most complex in optimal control theory. For quite a long time, from the moment of their occurrence and the first attempts at practical implementation in technical fields, these problems have been studied by experts from different angles. Much attention is traditionally paid to the development of computational methods for solving this class of problems in a wide area of applications [9,13]. At the same time, directions are being developed related to further generalizations and the development of the Pontryagin maximum principle [7,10], as well as with the extension of the classes of problem statements [8,12,14]. The questions of the existence, stability, optimality of solutions are studied in [11] and others.

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In our opinion, one of the most important areas of the theory of solving optimal control problems is the study of various approaches to the development of evidence-based methods for solving terminal control problems. The theory of evidence-based methods is currently an important tool in various application areas of mathematical modeling tools. In this theory, emphasis is placed on the ideas of proof, validity, and guarantee of the result. The latter assumes that the developed computing technology (computing process) generates a sequence of iterations that has a limit point on a bounded set, and this point is guaranteed to be a solution to the original problem with a given accuracy.

In this paper, we consider the problem of terminal control with phase constraints and their cross sections at discretization points. Intermediate spaces are associated with sampling points, the dimension of which is equal to the dimension of the phase trajectory vector. The sections of the phase trajectory in the spaces of sections form a polyhedral set. On this set, we pose the problem of minimizing a convex objective function. At each sampling point, we obtain some finite-dimensional problem. To iteratively proceed to the next phase trajectory, it is enough to take a gradient-type step in the section space for each intermediate problem. These steps together on all intermediate problems form a saddle-point gradient flow. This computational flow with an increase in the number of iterations leads us to the solution of the problem.

2 Statement of Terminal Control Problem with Continuous Phase Constraints

We consider a linear dynamic controlled system defined on a given time interval $[t_0, t_f]$, with a fixed left end and a moving right end under phase constraints on the trajectory. The dynamics of the controlled phase trajectory $x(t)$ is described by a linear system of differential equations with an implicit condition at the right end of the time interval. A terminal condition is defined as a solution to a linear programming problem that is not known in advance. In this case, it is necessary to choose a control so that the phase trajectory satisfies the phase constraints, and its right end coincides with the solution of the boundary value problem. The control problem is considered in a Hilbert function space.

Formally, everything said in the case of continuous phase constraints can be represented as a problem: find the optimal control $u(t) = u^*(t)$ and the corresponding trajectory $x(t) = x^*(t)$, $t \in [t_0, t_f]$, that satisfy the system

$$\begin{gathered}
\frac{d}{dt}x(t) = D(t)x(t) + B(t)u(t),\ t_0 \le t \le t_f,\ \ x(t_0) = x_0,\ x(t_f) = x_f^*, \\
G(t)x(t) \le g(t),\ x(t) \in \mathrm{R}^n\ \forall\, t \in [t_0, t_f], \\
u(t) \in \mathrm{U} = \{u(t) \in \mathrm{L}_2^r[t_0, t_f]\ \mid\ u(t) \in [u^-, u^+]\ \forall\, t \in [t_0, t_f]\}, \\
x_f^* \in \mathrm{Argmin}\{\langle \varphi_f, x_f\rangle \mid G_f x_f \le g_f,\ x_f \in \mathrm{R}^n\}, \qquad (1)
\end{gathered}$$

where $D(t), B(t), G(t)$ are continuous matrices of size $n \times n$, $n \times r$, $m \times n$ respectively; $g(t)$ is a given continuous vector function; $G_f = G(t_f)$, $g_f = g(t_f)$,

$x_f = x(t_f)$ are the values at the right-hand end of the time interval; φ_f is the given vector (normal to the linear objective functional), $x(t_0) = x_0$ is the given initial condition. The inclusion $x(t) \in \mathrm{R}^n$ means that the vector $x(t)$ for each t belongs to the finite-dimensional space R^n. The controls $u(t)$ for each $t \in [t_0, t_f]$ belong to the set U, which is a convex compact set from R^r. Problem (1) is considered as an analogue of the linear programming problem formulated in a functional Hilbert space.

To solve the differential system in (1), it is necessary to use the initial condition x_0 and some control $u(t) \in \mathrm{U}$. For each admissible $u(t)$, in the framework of the classical theorems of existence and uniqueness, we obtain a unique phase trajectory $x(t)$. The right end of the optimal trajectory must coincide with the finite-dimensional solution of the boundary value problem, i. e. $x^*(t_f) = x_f^*$. An asterisk means that $x^*(t)$ is the optimal solution; in particular, x_f^* is a solution to the boundary value optimization problem. The control must be selected so that phase constraints are additionally fulfilled. The left end x_0 of the trajectory is fixed and is not an object of optimization.

The formulated problem with phase constraints from the point of view of developing evidence-based computational methods is one of the difficult problems. Traditionally, optimal control problems (without a boundary value problem) are studied in the framework of the Hamiltonian formalism, the peak of which is the maximum principle. This principle is a necessary condition for optimality and is the dominant tool for the study of dynamic controlled problems. However, the maximum principle does not allow constructing methods that are guaranteed to give solutions with a predetermined accuracy. In the case of convex problems of type (1), it seems more reasonable to conduct a study in the framework of the Lagrangian formalism. Moreover, the class of convex problems in optimal control is wide enough, and almost any smooth problem can be approximated by a convex, quadratic, or linear problem.

Problem (1) without phase restrictions was investigated by the authors in [1–5]. In the linear-convex case, relying on the saddle-point inequalities of the Lagrange function, the authors proved the convergence of extragradient and extraproximal methods to solving the terminal control problem in all solution components: weak convergence in controls, strong convergence in phase and conjugate trajectories, and also in terminal variables of intermediate (boundary value) problems. This turned out to be possible due to the fact that the saddle-point inequalities for the Lagrange function in the case under consideration represent sufficient optimality conditions. These conditions, in contrast to the necessary conditions of the maximum principle, allow us to develop an evidence-based (without heuristic) theory of methods for solving optimal control problems, which was demonstrated in [1–5].

3 Phase Constraints Sections and Finite-Dimensional Intermediate Problems Generated by Them

In statement (1), we presented the phase constraints $G(t)x(t) \leq g(t)$, $t \in [t_0, t_f]$, of continuous type. An approach will be described below, where instead of con-

tinuous phase constraints their sections $G_s x_s \leq g_s$ are considered at certain instants of time t_s on a discrete grid

$$\Gamma = \{t_0, t_1, ..., t_{s-1}, t_s, t_{s+1}, ..., t_f\}.$$

At these moments of time, finite-dimensional cross-section problems are formed, and between these moments (on the discretization segments $[t_{s-1}, t_s]$, $s = \overline{1, f}$) intermediate terminal control problems arise. Thus, the original problem formulated on the entire segment $[t_0, t_f]$ is decomposed into a set of independent problems, each of which is defined on its own sub-segment. The obtained intermediate problems no longer have phase constraints, since the phase constraints on the sub-segments have passed to the boundary value problems at the ends of these sub-segments. This approach does not require the existence of a functional Slater condition. In fact, the existence of finite-dimensional saddle points in the intermediate spaces R^n (that are generated by sections of phase constraints at given moments t_s) is sufficient. Each section has its own boundary-value problem, and then through all these solutions (like a thread through a coal ear), the desired phase trajectory is drawn over the entire time interval. In finite-dimensional section spaces, the Slater condition for convex problems is always satisfied by definition.

Except for discrete phase constraints, the rest of problem (1) remains continuous. The combination of the trajectories and other components of the problem over all time sub-segments results in the solution of the original problem over the entire time interval $[t_0, t_f]$. The approach based on sections can be interpreted as a method of decomposing a complex problem into a number of simple ones.

Thus, on each of the segments $[t_{s-1}, t_s]$, a specific segment $x_s(t)$ of the phase trajectory of differential equation (1) is defined. At the common point of the adjacent segments $[t_{s-1}, t_s]$ and $[t_s, t_{s+1}]$ the values $x_s(t_s)$ and $x_{s+1}(t_s)$ coincide in construction: $x_s(t_s) = x_{s+1}(t_s)$, i.e. on each segment of the partition, the right end of the trajectory coincides with the starting point of the trajectory in the next segment.

As a result of discretization based on (1), the following statement of the multi-problem is obtained:

$$\begin{gathered}
\frac{d}{dt} x_s(t) = D(t) x_s(t) + B(t) u_s(t),\ t \in [t_{s-1}, t_s], \\
x_s(t_{s-1}) = x^*_{s-1}(t_{s-1}),\ x_s(t_s) = x^*_s,\ u_s(t) \in \mathrm{U}, \\
x^*_1 \in \mathrm{Argmin}\{\langle \varphi_1, x_1 \rangle \mid G_1 x_1 \leq g_1,\ x_1 \in \mathrm{R}^n\},\ x^*_1 \in X_1, \\
x^*_2 \in \mathrm{Argmin}\{\langle \varphi_2, x_2 \rangle \mid G_2 x_2 \leq g_2,\ x_2 \in \mathrm{R}^n\},\ x^*_2 \in X_2, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
x^*_f \in \mathrm{Argmin}\{\langle \varphi_f, x_f \rangle \mid G_f x_f \leq g_f\ x_f \in \mathrm{R}^n\},\ x^*_f \in X_f. \qquad (2)
\end{gathered}$$

Here $x_s(t_s)$ is the value of the function $x_s(t)$ at the right end of segment $[t_{s-1}, t_s]$, x^*_s is the solution of sth intermediate linear programming problem; φ_s is the normal to the objective function; X_s is an intermediate reachability set; $G_s = G(t_s)$, $g_s = g(t_s)$, $s = \overline{1, f}$. If we combine all parts of the trajectories $x_s(t)$ then

we get the full trajectory on the entire segment $x(t)$, $t \in [t_0, t_f]$. In other words, we "broke" the original problem (1) into f independent problems of the same kind.

For greater clarity, imagine system (2) in an expanded form. Discretization of Γ generates time intervals $[t_{s-1}, t_s]$, on which functions $x_s(t)$ are defined for all $s = \overline{1, f}$. Each of these functions is the restriction of the phase trajectory $x(t)$ to the segment $[t_{s-1}, t_s]$. In this model, for each sth time interval $[t_{s-1}, t_s]$, the sth controlled trajectory $x_s(t)$ and the sth intermediate problem are defined:

$$\begin{gathered}
\frac{d}{dt}x_1(t) = D(t)x_1(t) + B(t)u_1(t),\ t \in [t_0, t_1],\\
x_1(t_0) = x_0,\ x_1(t_1) = x_1^*,\ u_1(t) \in \mathrm{U},\\
x_1^* \in \mathrm{Argmin}\{\langle \varphi_1, x_1\rangle \mid G_1 x_1 \le g_1,\ x_1 \in \mathrm{R}^n\},\ x_1^* \in X_1, x_1(t_1) = x_1,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{d}{dt}x_s(t) = D(t)x_s(t) + B(t)u_s(t),\ t \in [t_{s-1}, t_s],\\
x_s(t_{s-1}) = x_{s-1}^*,\ x_s(t_s) = x_s^*,\ u_s(t) \in \mathrm{U},\\
x_s^* \in \mathrm{Argmin}\{\langle \varphi_s, x_s\rangle \mid G_s x_s \le g_s,\ x_s \in \mathrm{R}^n\},\ x_s^* \in X_s, x_s(t_s) = x_s, \qquad (3)\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{d}{dt}x_f(t) = D(t)x_f(t) + B(t)u_f(t),\ t \in [t_{f-1}, t_f],\\
x_f(t_{f-1}) = x_{f-1}^*,\ x_f(t_f) = x_f^*,\ u_f(t) \in \mathrm{U},\\
x_f^* \in \mathrm{Argmin}\{\langle \varphi_f, x_f\rangle \mid G_f x_f \le g_f,\ x_f \in \mathrm{R}^n\},\ x_f^* \in X_f, x_f(t_f) = x_f.
\end{gathered}$$

So, within the framework of the proposed approach, the initial problem with phase constraints (1) is split into a finite set of independent intermediate terminal control problems without phase constraints. Each of these problems can be solved independently, starting with the first problem. Then, conducting a phase trajectory through solutions of intermediate problems, we find a solution to the terminal control problem over the entire segment $[t_0, t_f]$. To solve any of the subproblems of system (3), the authors developed methods in [1,2].

4 Problem Statement in Vector-Matrix Form

For greater clarity, we present the system (2) or (3) in a more compact vector-matrix form:

dynamics

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_f}{dt} \end{pmatrix} = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_f \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_f \end{pmatrix} + \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_f \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_f \end{pmatrix}$$

where $x(t_0) = x_0$, $x_s(t_s) = x_s^*$, $x_f(t_f) = x_f^*$, $u_s(t) \in \mathrm{U}$,

and intermediate problems

$$\begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_f^* \end{pmatrix} \in \operatorname{Argmin} \left\{ \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_f \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_f \end{pmatrix} \middle| \begin{pmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_f \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_f \end{pmatrix} \leq \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_f \end{pmatrix} \right\} \tag{4}$$

Recall once again that each function $x(t)$ generates a vector with components $(x(t_1), ..., x(t_s), ..., x(t_f))$, and the number of components is equal to the number of sampling points of the segment $[t_0, t_f]$. Each component of this vector, in turn, is a vector of size n. Thus, we have a space of dimension $\mathrm{R}^{n\times f}$. In this space, the diagonal matrix $G(t_s)$, $s = \overline{1, f}$, is defined, each component of which is submatrix $G_s(t_s)$ from (4), whose dimension is $n \times n$. We described the matrix functional constraint of the inequality type at the right-hand side, which is given by vector $g = (g_1, g_2, ..., g_f)$. The linear objective function that completes the formulation of the finite-dimensional linear programming problem in (4) is a scalar product of vectors φ and x.

Thus, in macro format, we can represent problem (4) in the form

$$\begin{cases} \dfrac{d}{dt}x(t) = D(t)x(t) + B(t)u(t),\ t_0 \leq t \leq t_f,\ \ x(t_0) = x_0,\ x(t_f) = x_f^*, \\ \qquad\qquad x^* \in \operatorname{Argmin}\{\langle \varphi_f, x\rangle \mid Gx \leq g,\ x \in \mathrm{R}^n\}, u(t) \in \mathrm{U}. \end{cases} \tag{5}$$

Note that the macro system (5) obtained as a result of scalarization of intermediate problems (3) (or (4)) almost completely coincides with the terminal control problem with the boundary value problem on the right-hand end suggested and explored in [1,2]. Therefore, the method for solving problem (5) and the proof of its convergence as a whole will repeat the logic of reasoning.

As a solution to differential system (5), we mean any pair $(x(t), u(t)) \in \mathrm{L}_2^n[t_0, t_f] \times \mathrm{U}$ that satisfies the condition

$$x(t) = x(t_0) + \int_{t_0}^{t} (D(\tau)x(\tau) + B(\tau)u(\tau))d\tau, \quad t_0 \leq t \leq t_f. \tag{6}$$

The trajectory $x(t)$ in (6) is an absolutely continuous function. The class of absolutely continuous functions is a linear variety everywhere dense in $\mathrm{L}_2^n[t_0, t_f]$. In the future, this class will be denoted as $\mathrm{AC}^n[t_0, t_f] \subset \mathrm{L}_2^n[t_0, t_f]$. For any pair of functions $(x(t), u(t)) \in \mathrm{AC}^n[t_0, t_f] \times \mathrm{U}$, the Newton-Leibniz formula and, accordingly, the integration-by-parts formula are satisfied.

5 Saddle-Point Sufficient Optimality Conditions. Dual Approach

Drawing the corresponding analogies with the theory of linear programming, we write out the primal and dual Lagrange functions for the problem (5). To do

this, we scalarize system (5) and introduce a linear convolution known as the Lagrange function

$$\mathcal{L}(p,\psi(t);x,x(t),u(t)) = \langle\varphi,x\rangle + \langle p, Gx-g\rangle$$
$$+\int_{t_0}^{t_f}\langle\psi(t), D(t)x(t)+B(t)u(t)-\frac{d}{dt}x(t)\rangle dt,$$

defined for all $p\in \mathrm{R}^m_+$, $\psi(t)\in \Psi^n_2[t_0,t_f]$, $x\in \mathrm{R}^n$, $(x(t),u(t))\in \mathrm{AC}^n[t_0,t_f]\times \mathrm{U}$. Here x is a finite-dimensional vector composed of the values of trajectory $x(t)$ at the sampling points; $\Psi^n_2[t_0,t_f]$ is a linear manifold of absolutely continuous functions from an adjoint space. The variety $\Psi^n_2[t_0,t_f]$ is everywhere dense in $\mathrm{L}^n_2[t_0,t_f]$.

The saddle point $(p^*,\psi^*(t);x^*,x^*(t),u^*(t))$ of the Lagrange function is formed by primal $(x^*,x^*(t),u^*(t))$ and dual $(p^*,\psi^*(t))$ solutions of problem (5) and, by definition, satisfies the system of inequalities

$$\langle\varphi,x^*\rangle + \langle p, Gx^*-g\rangle + \int_{t_0}^{t_f}\langle\psi(t), D(t)x^*(t)+B(t)u^*(t)-\frac{d}{dt}x^*(t)\rangle dt$$
$$\leq \langle\varphi,x^*\rangle + \langle p^*, Gx^*-g\rangle + \int_{t_0}^{t_f}\langle\psi^*(t), D(t)x^*(t)+B(t)u^*(t)-\frac{d}{dt}x^*(t)\rangle dt$$
$$\leq \langle\varphi,x\rangle + \langle p^*, Gx-g\rangle + \int_{t_0}^{t_f}\langle\psi^*(t), D(t)x(t)+B(t)u(t)-\frac{d}{dt}x(t)\rangle dt$$

for all $p\in \mathrm{R}^m_+$, $\psi(t)\in \Psi^n_2[t_0,t_f]$, $x\in \mathrm{R}^n$, $(x(t),u(t))\in \mathrm{AC}^n[t_0,t_f]\times \mathrm{U}$.

If the original problem (5) has a primal and dual solution, then this pair is a saddle point of the Lagrange function. Here, as in the finite-dimensional case, the dual solution is formed by the coordinates of the normal to the supporting plane at the minimum point.

The converse is also true: the saddle point of the Lagrange function consists of the primal and dual solutions to original problem (5).

Using formulas to go over to conjugate linear operators

$$\langle\psi, Dx\rangle = \langle D^T\psi, x\rangle,\ \langle\psi, Bu\rangle = \langle B^T\psi, u\rangle$$

and the integrating-by-parts formula on segment $[t_0,t_f]$

$$\langle\psi(t_f),x(t_f)\rangle - \langle\psi(t_0),x(t_0)\rangle = \int_{t_0}^{t_f}\langle\frac{d}{dt}\psi(t),x(t)\rangle dt + \int_{t_0}^{t_f}\langle\psi(t),\frac{d}{dt}x(t)\rangle dt,$$

we write out the dual Lagrange function and saddle-point system in the conjugate form:

$$\mathcal{L}^T(p,\psi(t);x,x(t),u(t)) = \langle\varphi+G^Tp-\psi_f,x\rangle - \langle g,p\rangle + \langle\psi_0,x_0\rangle$$
$$+\int_{t_0}^{t_f}\langle D^T(t)\psi(t)+\frac{d}{dt}\psi(t),x(t)\rangle dt + \int_{t_0}^{t_f}\langle B^T(t)\psi(t),u(t)\rangle dt$$

for all $p \in \mathrm{R}^m_+$, $\psi(t) \in \Psi^n_2[t_0, t_f]$, $x \in \mathrm{R}^n$, $(x(t), u(t)) \in \mathrm{AC}^n[t_0, t_f] \times \mathrm{U}$, $x_0 = x(t_0)$, $\psi_0 = \psi(t_0)$, $\psi_f = \psi(t_f)$.

The dual saddle-point system has the form

$$\langle \varphi + G^T p - \psi_f, x^* \rangle + \langle -g, p \rangle + \langle \psi_0, x^*_0 \rangle$$

$$+ \int_{t_0}^{t_f} \langle D^T(t)\psi(t) + \tfrac{d}{dt}\psi(t), x^*(t) \rangle dt + \int_{t_0}^{t_f} \langle B^T(t)\psi(t), u^*(t) \rangle dt \leq$$

$$\leq \langle \varphi + G^T p^* - \psi^*_f, x^* \rangle + \langle -g, p^* \rangle + \langle \psi^*_0, x^*_0 \rangle$$

$$+ \int_{t_0}^{t_f} \langle D^T(t)\psi^*(t) + \tfrac{d}{dt}\psi^*(t), x^*(t) \rangle dt + \int_{t_0}^{t_f} \langle B^T(t)\psi^*(t), u^*(t) \rangle dt \leq$$

$$\leq \langle \varphi + G^T p^* - \psi^*_f, x \rangle + \langle -g, p^* \rangle + \langle \psi^*_0, x_0 \rangle$$

$$+ \int_{t_0}^{t_f} \langle D^T(t)\psi^*(t) + \tfrac{d}{dt}\psi^*(t), x(t) \rangle dt + \int_{t_0}^{t_f} \langle B^T(t)\psi^*(t), u(t) \rangle dt$$

for all $p \in \mathrm{R}^m_+$, $\psi(t) \in \Psi^n_2[t_0, t_1]$, $x \in \mathrm{R}^n$, $(x(t), u(t)) \in \mathrm{AC}^n[t_0, t_f] \times \mathrm{U}$.

Both Lagrangians (primal and dual) have the same saddle point $(p^*, \psi^*(t); x^*, x^*(t), u^*(t))$, which satisfies the saddle-point conjugate system.

From the analysis of the saddle-point inequalities, we can write out mutually dual problems:

the primal problem:

$$x^* \in \mathrm{Argmin}\{\langle \varphi, x \rangle \mid Gx \leq g,\ x \in \mathrm{R}^n,$$

$$\tfrac{d}{dt}x(t) = D(t)x(t) + B(t)u(t), \quad x(t_0) = x_0,\ u(t) \in \mathrm{U}\};$$

the dual problem:

$$(p^*, \psi^*(t)) \in \mathrm{Argmax}\{\langle -g, p \rangle + \langle \psi_0, x^*_0 \rangle + \int_{t_0}^{t_f} \langle \psi(t), B(t)u^*(t) \rangle dt \ \Big|$$

$$D^T(t)\psi(t) + \tfrac{d}{dt}\psi(t) = 0, \ \ \psi_f = \varphi + G^T p,$$

$$p \in \mathrm{R}^m_+,\ \psi(t) \in \Psi^n_2[t_0, t_f]\},$$

$$\int_{t_0}^{t_f} \langle B^T(t)\psi^*(t), u^*(t) - u(t) \rangle dt \leq 0, \ \ u(t) \in \mathrm{U}.$$

6 Method for Solving. Convergence Technique

Replacing the variational inequalities in the above system with the corresponding equations with the projection operator, we can write the differential system in operator form. Then, based on this system, we write out a saddle-point method of extragradient type to calculate the saddle point of the Lagrange function. The two components of the saddle point are the primal and dual solutions to problem (5).

The formulas of this iterative method are as follows:

1) predictive half-step

$$\tfrac{d}{dt}x^k(t) = D(t)x^k(t) + B(t)u^k(t),\ x^k(t_0) = x_0,$$

$$\bar{p}^k = \pi_+(p^k + \alpha(Gx^k - g)),$$

$$\tfrac{d}{dt}\psi^k(t) + D^T(t)\psi^k(t) = 0,\ \psi^k = \varphi + G^T p^k,$$

$$\bar{u}^k(t) = \pi_U(u^k(t) - \alpha B^T(t)\psi^k(t));$$

2) basic half-step

$$\tfrac{d}{dt}\bar{x}^k(t) = D(t)\bar{x}^k(t) + B(t)\bar{u}^k(t),\ \bar{x}^k(t_0) = x_0,$$

$$p^{k+1} = \pi_+(p^k + \alpha(G\bar{x}^k - g)),$$

$$\tfrac{d}{dt}\bar{\psi}^k(t) + D^T(t)\bar{\psi}^k(t) = 0,\ \bar{\psi}^k = \varphi + G^T\bar{p}^k,$$

$$u^{k+1}(t) = \pi_U(u^k(t) - \alpha B^T(t)\bar{\psi}^k(t)),\ k = 0, 1, 2, ...$$

Here, at each half-step, two differential equations are solved and an iterative step along the controls is carried out. Below, a theorem on the convergence of the method to the solution is formulated.

Theorem 1. *If the set of solutions* $(p^*, \psi^*(t); x^*, x^*(t), u^*(t))$ *for problem (5) is not empty, then sequence* $\{(p^k, \psi^k(t); x^k, x^k(t), u^k(t))\}$ *generated by the method with the step length* $\alpha \leq \alpha_0$ *contains subsequence* $\{(p^{k_i},\ \psi^{k_i}(t);\ x^{k_i},\ x^{k_i}(t), u^{k_i}(t))\}$*, which converges to the solution of the problem, including: weak convergence in controls, strong convergence in trajectories, conjugate trajectories, and also in terminal variables.*

The proof of the theorem is carried out in the same way as in [6]. The computational process presented in this paper implements the idea of evidence-based computing. It allows us to receive guaranteed solutions to the problem with a given accuracy, consistent with the accuracy of the initial information.

Conclusions. In this paper, we study a terminal control problem with a finite-dimensional boundary value problem at the right-hand end of the time interval and phase constraints distributed over a finite given number of points of this interval. The problem has a convex structure, which makes it possible, within the duality theory, using the saddle-point properties of the Lagrangian, to develop a theory of saddle-point methods for solving terminal control problems. The approach proposed here makes it possible to deal with a complex case with intermediate phase constraints on the controlled phase trajectories. The convergence of the computation process for all components of the solution has been proved: namely, weak convergence in controls and strong convergence in phase and dual trajectories and in terminal variables.

References

1. Antipin, A.S., Khoroshilova, E.V.: Linear programming and dynamics. Ural Math. J. **1**(1), 3–19 (2015)
2. Antipin, A.S., Khoroshilova, E.V.: Saddle-point approach to solving problem of optimal control with fixed ends. J. Global Optim. **65**(1), 3–17 (2016)
3. Antipin, A.S., Khoroshilova, E.V.: On methods of terminal control with boundary value problems: Lagrange approach. In: Goldengorin, B. (ed.) Optimization and Applications in Control and Data Sciences, pp. 17–49. Springer, New York (2016)
4. Antipin, A.S., Khoroshilova, E.V.: Lagrangian as a tool for solving linear optimal control problems with state constraints. In: Proceedings of the International Conference on Optimal Control and Differential Games Dedicated to L.S. Pontryagin on the Occasion of His 110th Birthday, pp. 23–26 (2018)
5. Antipin, A., Khoroshilova, E.: Controlled dynamic model with boundary-value problem of minimizing a sensitivity function. Optim. Lett. **13**(3), 451–473 (2017). https://doi.org/10.1007/s11590-017-1216-8
6. Antipin, A.S., Khoroshilova, E.V.: Dynamics, phase constraints, and linear programming. Comput. Math. Math. Phys. **60**(2), 184–202 (2020)
7. Dmitruk, A.V.: Maximum principle for the general optimal control problem with phase and regular mixed constraints. Comput. Math. Model **4**, 364–377 (1993)
8. Dykhta, V., Samsonyuk, O.: Some applications of Hamilton-Jacobi inequalities for classical and impulsive optimal control problems. Eur. J. Control **17**(1), 55–69 (2011)
9. Gornov, A.Y., Tyatyushkin, A.I., Finkelstein, E.A.: Numerical methods for solving terminal optimal control problems. Comput. Math. Math. Phys. **56**(2), 221–234 (2016)
10. Hartl, R.F., Sethi, S.P., Vickson, R.G.: A survey of the maximum principles for optimal control problems with state constraints. SIAM Rev. **37**(2), 181–218 (1995)
11. Mayne, D.Q., Rawlings, J.B., Rao, C.V., Scokaert, P.O.M.: Survey constrained model predictive control: stability and optimality. Automatica (J. IFAC) **36**(6), 789–814 (2001)
12. Pales, Z., Zeidan, V.: Optimal control problems with set-valued control and state constraints. SIAM J. Optim. **14**(2), 334–358 (2003)
13. Pytlak, R.: Numerical Methods for Optimal Control Problems with State Constraints. Lecture Notes in Mathematics, vol. 1707, p. 214p. Springer, Berlin, Heidelberg (1999)
14. Rao, A.V.: A Survey of Numerical Methods for Optimal Control. Advances in the Astronautical Sciences. Preprint AAS 09-334 (2009)