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Nonintegrable threshold singularities of two-point functions in perturbation theory

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Abstract In perturbation theory, the spectral densities of two-point functions develop nonintegrable threshold singularities at higher orders. In QCD, such singularities emerge when calculating the diagrams in terms of the pole quark mass, and they become stronger when one rearranges the perturbative expansion in terms of the running quark mass. In this paper, we discuss the proper way to handle such singularities.

1 Introduction

The correlation function of two currents defined as

$$\Pi(q) = i \int \mathrm{d}x e^{iqx} \langle T\{\bar{\Psi}(x)\mathcal{O}_1\psi(x), \bar{\psi}(0)\mathcal{O}_2\Psi(0)\}\rangle,\tag{1.1}$$

where ψ and Ψ denote fermion fields (which may be identical) and $\mathcal{O}_{1,2}$ are Dirac matrices, is one of the basic objects in quantum field theory. For instance, in QCD, two-point functions with an appropriate choice of the Dirac matrices provide the basis for the extraction of masses and couplings of mesons with the corresponding quantum numbers within the method of QCD sum rules [1,2]. In general, the two-point function contains a number of independent Lorentz structures $L_i(q)$ and the corresponding invariant amplitudes $F_i(q^2)$. We shall discuss here spectral representations for the invariant amplitudes $F_i(q^2)$ and omit throughout the paper the subscript *i*.

As follows from the general properties of the time-ordered product, the function $F(q^2)$ is an analytic function in the complex q^2 -plane with the cut along the real axis [3,4] from a threshold s_{th} to $+\infty$ and satisfies the spectral representation with an appropriate number n of subtractions:

$$F(q^2) = F(0) + F'(0)q^2$$

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$$+\dots + \frac{(q^2)^n}{\pi} \int_{s_{\rm th}}^{\infty} \frac{\mathrm{d}s}{s^n(s-q^2-i0)} \rho(s), \qquad (1.2)$$

where $\rho(s) = \text{Im } F(s + i0)$. The subtractions are performed in order to provide the convergence of the spectral integral or to satisfy constraints imposed by symmetries of the theory.

In QCD, one makes use of several expansions of the two-point functions. Important examples of such expansions are listed below: (i) perturbative expansion in powers of α_s of the elastic correlation function [5–10]; (ii) rearrangement of the perturbation theory for the heavy-light correlation functions [11, 12] via a replacement of the pole mass by the running mass of the heavy quark [13–15] to gain better convergence of the perturbation series; (iii) expansion of the heavy-light correlation functions in the light-quark mass [16, 17]. In all these cases, higher-order spectral densities exhibit the appearance of nonintegrable divergences at the threshold.

Reference [10] gave a detailed analysis of the dispersion representations in the presence of threshold singularities, with the emphasis on vacuum polarization in QED: The authors made use of the known analytic results for the elastic QED two-point function (including the contributions of the continuum and the positronium poles) and its expansion in perturbation theory, in order to demonstrate their equivalence in the framework of dispersion relations.

The novel feature of this paper is the procedure for handling nonintegrable threshold singularities in a more general case when only the discontinuity of the analytic function on the cut, but not the function itself, is known. We formulate an algorithm for solving the integral equation for the analytic function which arises when its discontinuity contains nonintegrable threshold singularities. Our analysis covers all three cases (i)–(iii) mentioned above and may be easily generalized to a nonintegrable threshold singularity of any form.

2 Threshold singularities in perturbation theory

As an example consider radiative QED corrections to vacuum polarization in the vicinity of the e^+e^- threshold. The continuum contribution to the spectral density of the vacuum polatization is known exactly and is given by the so-called Sommerfeld factor [2]:

$$\rho(s) = \frac{\pi\alpha}{2} \frac{1}{1 - \exp(-\pi\alpha/\nu)}, \quad \nu = \sqrt{1 - \frac{4m^2}{s}},$$
(2.1)

m the electron mass and α the electromagnetic coupling. Expanding this factor in powers of α yields

$$\rho(s) = \frac{v}{2} + \frac{\pi\alpha}{4} + \frac{1}{2} \sum_{n=2,4\dots} \frac{B_n}{n!} (\pi\alpha)^n \frac{1}{v^{n-1}},$$
(2.2)

where B_n are Bernoulli numbers ($B_2 = 1/6$, $B_4 = -1/30$, ...). The sum in Eq. (2.2) runs over even powers of α : for n = 2 it has an integrable 1/v divergence at the threshold v = 0, but starting with n = 4, these singularities are nonintegrable and one should specify the precise way to handle them.¹

¹ In addition to the continuum contribution, there is also the contribution of positronium poles located below the threshold (see, e.g., [2]). Expanding the latter in powers of α and adding to the perturbative expansion of the continuum contribution, one gets the full perturbative spectral density. For our interest in this paper, only the perturbative expansion of the continuum is relevant: The contributions of the positronium poles are localized

Fig. 1 Deformations of the integration contour $\Gamma_1 \rightarrow \Gamma_2 \rightarrow \Gamma_3$ in Eq. (2.3). According to the Cauchy theorem, such deformations of the contour do not change the finite value of the contour integral



To understand the proper way to proceed, we recall the following property of any analytic function: A contour integral of an analytic function over the region where the function is free of singularities is finite and does not depend on the specific choice of the integration contour. Let us look at the problem of the threshold divergences from this perspective. We start with the Cauchy theorem

$$F(s) = F(0) + \frac{s}{2\pi i} \oint_{\Gamma} \frac{F(s')}{s'(s'-s)} ds'.$$
 (2.3)

Here Γ is any contour surrounding the point *s* and located in the region where the function F(s) is analytic. We start with the contour Γ_1 (see Fig. 1). Obviously, the integral is finite. We now start to deform the contour $\Gamma_1 \rightarrow \Gamma_2 \rightarrow \Gamma_3$ in the region of analyticity of the function F(s). Such deformations do not change the (finite) value of the integral. Finally, we end up with the contour Γ_3 which embraces the cut from $s = 4m^2$ to $s = \infty$, and the large circle. With the appropriate number of subtractions, the large-circle integral vanishes, and we end up with the contour integral which embraces the cut. This integral needs some care. It may be split into two parts: (I) the integral over a small circle of radius ϵ around the threshold $s = 4m^2$, and (II) the integral over the "cut" starting from $s = 4m^2 + \epsilon$ to ∞ along the real *s*-axis. The sum of these two integrals is finite due to the general property of the analytic functions mentioned above. If the behavior of the function F(s) near the threshold is such that the integral of Im F(s) over the cut is finite, then the small-circle integral vanished as $\epsilon \to 0$. However, the situation changes for the case when Im F(s) has a nonintegrable singularity at the threshold: then, the small-circle integral (I) also diverges for $\epsilon \to 0$, making the sum of (I) and (II) finite for $\epsilon \to 0$.

We shall demonstrate this property for the general case of the singular threshold behavior of the spectral density of the form Im $F(s + i0) = \rho(s) \simeq 1/v^n$. According to the behavior of the imaginary part of F(s), Im $F(s) = \rho(s) \rightarrow 1$ at large *s*, one subtraction in the spectral representation for F(s) is sufficient to guarantee that the large-contour integral vanishes.

slightly below the threshold and fall on the threshold being expanded in powers of α . Such contributions are integrable at the threshold and therefore are beyond the scope of our interest. To avoid confusion, we only recall that the positronium poles lead to nonzero small-circle contributions [10] [see Eq. (2.4)].

For convenience, we introduce a dimensionless variable $z = s/4m^2$, such that the cut in F(z) is located along the real z-axis from 1 to ∞ , and write for the function F(z) the following dispersion representation with one subtraction (see also [10]):

$$F(z) = F(0) + \frac{z}{2\pi i} \oint_{R_{\epsilon}} \frac{F(z')}{z'(z'-z)} dz' + \frac{z}{\pi} \int_{1+\epsilon}^{\infty} \frac{\rho(z')}{z'(z'-z-i0)} dz'.$$
 (2.4)

where R_{ϵ} is the (clockwise) circle with radius ϵ and the center at z = 1. Equation (2.4) is an inhomogeneous integral equation for F(z) and its solution is not fully trivial.

Technically, the cases of an even *n* and an odd *n* are slightly different, so we shall consider them separately.

2.1 Odd powers of 1/v

The spectral density of the elastic two-point function, Eq. (2.2), contains only odd inverse powers of v. Let us start with the case of $O(\alpha_s^4)$ correction, namely

$$\rho_4(s) \propto \frac{1}{v^3}, \quad v = \sqrt{1 - \frac{4m^2}{s}}.$$
(2.5)

A detailed discussion of the above correction in the case of the hadronic vacuum polarization contribution to the electron (g - 2) can be found in Ref. [10].

We first analyze the contribution to $F_{3/2}(z)$ from the dispersion integral over the cut and thus see which parts of this contribution are to be inserted in the small-circle integral. By a manipulation with the integrand,

$$\frac{1}{1 - \frac{z}{z'}} = \left(\frac{1}{1 - \frac{z}{z'}} - \frac{1}{1 - z}\right) + \frac{1}{1 - z}$$
$$= -\frac{z}{1 - z}\frac{1 - \frac{1}{z'}}{1 - \frac{z}{z'}} + \frac{1}{1 - z},$$
(2.6)

we isolate the term singular in ϵ and obtain

$$F_{3/2}(z,\epsilon) \equiv \frac{z}{\pi} \int_{1+\epsilon}^{\infty} \frac{1}{(1-1/z')^{3/2} z'(z'-z)} dz'$$

= $\frac{2z}{\pi(1-z)} \frac{1}{\sqrt{\epsilon}} - \frac{2z}{\pi(1-z)} - \frac{z}{1-z} F_{1/2}(z,0)$
+ $O(\sqrt{\epsilon}),$ (2.7)

where

$$F_{1/2}(z,\epsilon) \equiv \frac{z}{\pi} \int_{1+\epsilon}^{\infty} \frac{1}{(1-1/z')^{1/2} z'(z'-z-i0)} dz',$$

$$F_{1/2}(z,0) = -\frac{2}{\pi} \sqrt{\frac{z}{z-1}} \log\left(\sqrt{-z} + \sqrt{1-z}\right).$$
(2.8)

It is convenient to write (2.7) in the following form

$$F_{3/2}(z,\epsilon) = \frac{2}{\pi} \frac{z}{1-z} \frac{1}{\sqrt{\epsilon}} + \frac{2}{\pi} \frac{z}{z-1} \left[1 - \sqrt{\frac{z}{z-1}} \log\left(\sqrt{z} + \sqrt{z-1}\right) \right] - \left(\frac{z}{1-z}\right)^{3/2} + O(\sqrt{\epsilon}),$$
(2.9)

where the second term on the r.h.s. is finite at the threshold.

Let us write down again Eq. (2.4):

$$F_{3/2}(z) = F_{3/2}(0) + \frac{z}{2\pi i} \oint_{R_{\epsilon}} \frac{F_{3/2}(z')}{z'(z'-z)} dz' + F_{3/2}(z,\epsilon).$$
(2.10)

Because of the Cauchy theorem, the function $F_{3/2}(z)$ does not depend on ϵ . This means that the term $\sim \epsilon^{-1/2}$ generated by $F_{3/2}(z, \epsilon)$ cannot be a part of $F_{3/2}(z)$ and should cancel against the small-circle integral. We shall now demonstrate that the cancellation of this divergent term is the only effect of the small-circle integral and that it does not yield any contribution that remains finite in the limit $\epsilon \rightarrow 0$. In other words, *to obtain* $F_{3/2}(z)$ *one needs to subtract from* $F_{3/2}(z, \epsilon)$ *all singular terms in* ϵ *and then send* $\epsilon \rightarrow 0$. To prove this statement, we just show that $F_{3/2}(z)$ obtained in this way satisfies Eq. (2.10).

We turn to Eq. (2.9) and omit the first term in its r.h.s., which is singular in ϵ . Then we check which of the remaining structures may give a nonvanishing contribution when integrated over the small circle. The second term in the r.h.s. of Eq. (2.9) is nonsingular at the threshold, as it can be easily checked by an expansion around z = 1. Therefore, its contribution to the small-circle integral vanishes after the limit $\epsilon \rightarrow 0$ is taken.

Only the last term in the r.h.s. of Eq. (2.9), which is singular at the threshold, can contribute to the small-circle integral. It is convenient to set $z' = 1 - \epsilon e^{i\phi}$ and take the ϕ -integral from π to $-\pi$ that corresponds to the clockwise contour integration (the angle ϕ is measured from the negative direction of the z-axis; at $\phi = -\pi$, we are on the upper boundary of the cut, and at $\phi = \pi$ we are on its lower boundary), one gets

$$I_{3/2}(z,\epsilon) \equiv -\frac{z}{2\pi i} \oint_{R_{\epsilon}} \frac{\sqrt{z'}}{(1-z')^{3/2}} \frac{\mathrm{d}z'}{z'-z}$$
$$= \frac{z}{2\pi (1-z)\sqrt{\epsilon}} \int_{-\pi}^{\pi} \mathrm{d}\phi \ \mathrm{e}^{-i\phi/2} + O(\sqrt{\epsilon})$$
$$= -\frac{2}{\pi} \frac{z}{1-z} \frac{1}{\sqrt{\epsilon}} + O(\sqrt{\epsilon}), \qquad (2.11)$$

where we made use of the integral $\int_{-\pi}^{\pi} d\phi \, e^{-i\phi/2} = 4$. Thus, the small-circle integral $I_{3/2}(z, \epsilon)$ precisely cancels the singular contribution in $1/\sqrt{\epsilon}$ coming from $F_{3/2}(z, \epsilon)$ and does not develop any finite terms in the limit $\epsilon \to 0$.

Finally, in the case of the perturbative term (2.5), we come to the following solution for $F_{3/2}(z)$:

$$F_{3/2}(z) = F_{3/2}(0) + \frac{2}{\pi} \frac{z}{z-1} - \frac{2}{\pi} \frac{z}{z-1} \sqrt{\frac{z}{z-1}} \log(\sqrt{-z} + \sqrt{1-z}).$$
(2.12)

It is straightforward to check that on the upper boundary of the cut along the real axis from z = 1 to $+\infty$ [to get there one needs to set $z \rightarrow z + i0$ for z > 1], the imaginary part of F(z) is indeed equal to $(1 - 1/z)^{-3/2}$. The behavior of $F_{3/2}(z)$ near the threshold z = 1 reads

$$F_{3/2}(z+i0) - F_{3/2}(0) = i\left(\frac{z}{z-1}\right)^{3/2} - \frac{2}{3\pi} - \frac{2}{5\pi}(z-1) + O((z-1)^2),$$
(2.13)

and $F_{3/2}(z)$ verifies Eq. (2.4) in the limit $\epsilon \to 0$.

For higher-order contributions, the small-circle integrals produce singular terms containing a series of inverse powers of $\sqrt{\epsilon}$ related to poles of the increasing order at the threshold z = 1.

A recursive expression for $F_{n+\frac{1}{2}}(z)$ for an arbitrary integer $n \ge 1$ reads (a derivation of this expression and a proof by induction of the cancellation of all ϵ -dependent terms is given in Appendix A):

$$F_{n+\frac{1}{2}}(z) - F_{n+\frac{1}{2}}(0) = \frac{z}{z-1} \left\{ F_{n-\frac{1}{2}}(z) - F_{n-\frac{1}{2}}(0) \right\} + \frac{1}{\pi} \frac{z}{z-1} \frac{1}{(n-\frac{1}{2})}.$$
(2.14)

2.2 Even powers of 1/v

In several practical applications, e.g., when considering the expansion of the heavy–light twopoint correlation functions, one encounters spectral densities with threshold singularities of the type $1/v^{2n}$, with integer n [16].² Conceptually, this case has no difference compared to the odd powers considered previously, but technically there are some subtleties.

The dispersion integral over the cut reads

$$F_n(z,\epsilon) = \frac{z}{\pi} \int_{1+\epsilon}^{\infty} dz' \frac{1}{(1-\frac{1}{z'})^n} \frac{1}{z'(z'-z)}.$$
(2.15)

Using Eq. (2.6), one can obtain the following relation

$$F_n(z,\epsilon) = \frac{z}{z-1} F_{n-1}(z,\epsilon) -\frac{1}{\pi} \frac{z}{z-1} \int_{1+\epsilon}^{\infty} dz' \frac{1}{(1-\frac{1}{z'})^n} \frac{1}{z'^2}, \qquad (2.16)$$

² One should not confuse even powers of 1/v in the spectral density of the two-point function considered here and even powers of 1/v in the two-point function itself [10]: the latter emerge in the elastic two-point function in QED and correspond to the δ -function and its derivatives in the spectral density and therefore do not represent nonintegrable threshold singularities.

that gives, after taking the integral,

$$F_n(z,\epsilon) = \frac{z}{z-1} F_{n-1}(z,\epsilon) -\frac{1}{\pi} \frac{z}{z-1} \frac{\left[(1+\frac{1}{\epsilon})^n - 1\right]\epsilon - 1}{(n-1)(1+\epsilon)}.$$
 (2.17)

For n > 1, the last term contains only positive powers of $1/\epsilon$ (in particular, no constant term). As already explained above, the ϵ -dependent piece of $F_n(z, \epsilon)$ cannot be a part of the analytic function $F_n(z)$. So, (2.17) suggests that the analytic function $F_n(z)$ satisfies the relation

$$F_n(z) - F_n(0) = \frac{z}{z-1} \left\{ F_{n-1}(z) - F_{n-1}(0) \right\}.$$
 (2.18)

For n = 1, one finds

$$F_1(z,\epsilon) = -\frac{1}{\pi} \frac{z}{z-1} \log(1-z) + \frac{1}{\pi} \frac{z}{z-1} \log(\epsilon) + O(\epsilon)$$
(2.19)

and its ϵ -independent part suggests

$$F_1(z) - F_1(0) = -\frac{1}{\pi} \frac{z}{z-1} \log(1-z).$$
(2.20)

Starting with this function, Eq. (2.18) yields

$$F_n(z) - F_n(0) = -\frac{1}{\pi} \left(\frac{z}{z-1}\right)^n \log(1-z).$$
(2.21)

This solution is still our conjecture and we need to check that $F_n(z)$ given by (2.21) satisfies the integral equation (2.6). To this end, we insert (2.21) in the small-circle integral and check that the only role of the small-circle integral is indeed to precisely cancel the singular ϵ -dependent terms in $F_n(z, \epsilon)$.

Let us illustrate how things work, e.g., for n = 2:

$$F_2(z,\epsilon) = -\frac{1}{\pi} \left(\frac{z}{z-1}\right)^2 \log(1-z) + \frac{1}{\pi} \left(\frac{z}{z-1}\right)^2 \log(\epsilon) + \frac{1}{\pi} \frac{z}{1-z} \frac{1}{\epsilon} + O(\epsilon).$$
(2.22)

We now have to insert its ϵ -independent part, which as expected gives the solution $F_2(z)$, in the contour integral. Substituting $z' = 1 - \epsilon \exp(i\phi)$, such that the clockwise ϕ integral runs from π to $-\pi$, we obtain

$$I_{2}(z,\epsilon) = -\frac{z}{1-z} \times \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{\pi} \frac{\epsilon i e^{i\phi}}{\epsilon^{2} e^{2i\phi}} \frac{1-\epsilon e^{i\phi}}{1-\epsilon e^{i\phi}\frac{1}{1-z}} \left[\log(\epsilon)+i\phi\right]$$
(2.23)

yielding

$$I_2(z,\epsilon) = -\frac{1}{\pi} \left(\frac{z}{z-1}\right)^2 \log(\epsilon) - \frac{1}{\pi} \frac{z}{1-z} \frac{1}{\epsilon} + O(\epsilon).$$
(2.24)

The small-circle integral cancels all ϵ -singular terms in the integral over the cut and does not add any finite contributions for $\epsilon \to 0$.

A proof by induction of the cancellation of the ϵ -dependent terms (not only singular but all ϵ -dependent terms) in the sum $F_n(z, \epsilon) + I_n(z, \epsilon)$ for any integer *n* is given in Appendix B. So, our procedure indeed yields the solution of the integral equation (2.4) in the form (2.21).

Before closing this Section, we emphasize that it is not necessary to calculate the smallcircle integrals explicitly; it is sufficient to calculate the integral over the cut in Eq. (2.4): The ϵ -independent part gives the analytic function F(z), while (due to the Cauchy theorem) the ϵ -dependent terms in the dispersion integral in Eq. (2.4) are precisely cancelled by the small-circle contribution. As we have seen, the cancellation of the ϵ -dependent terms is the only role of the small-circle integral in the case of nonintegrable threshold singularities; it does not provide any finite contributions in the limit $\epsilon \rightarrow 0$.

3 Conclusions

We discussed the way to handle properly the threshold divergences arising in perturbation theory for two-point functions. Our results are as follows:

(i) Taking a proper account of the small-circle integral around the threshold leads to the "surface term" that exactly cancels the threshold divergence of the spectral integral. Thus, a properly defined dispersion representation for F(s) (and in fact for any analytic function) at each order of perturbative expansion is finite and does not have any threshold divergence.

It should be understood that when the small-circle integral provides a nonzero contribution, the Cauchy theorem leads to the integral equation for the function F(s) even if its discontinuity $\rho(s)$ on the cut is known analytically. We formulated an algorithm for solving this integral equation: Namely, one should calculate the integral over the cut and isolate in this integral the ϵ -singular and the ϵ -regular parts; the ϵ -regular part gives the desired solution of the integral equation, while the ϵ -singular part precisely cancels the small-circle integral. Noteworthy, the only role of the small-circle integral is the cancellation of the ϵ -singularities. At each order of the perturbative expansion, the small-circle integral does not lead to finite contributions.

We presented the analysis of nonintegrable threshold divergences of the general structure $\rho(s) \sim 1/v^n$, where *n* can be both even and odd integer numbers $n \ge 2$. Moreover, it should be clear that the proposed algorithm is applicable to any singular threshold behavior of the spectral density.

Let us emphasize that thanks to the Cauchy theorem the explicit calculation of the smallcircle integrals is not required. It is sufficient to isolate all terms singular in ϵ in the dispersion integral of $\rho(s)$: The small-circle integral is precisely equal to the singular part of the integral over the cut taken with an opposite sign. Noteworthy, the small-circle contribution does not yield any finite remnants in the limit $\epsilon \rightarrow 0$.

(ii) We would like to mention once more a detailed analysis of the threshold singularities in the dispersion representation for the vacuum polarization in QED [10], in which case the spectral function contains contributions of the continuum states and the bound positronium states. The analysis of [10] made use of the knowledge of the analytic results of both the vacuum polarization and its imaginary part to demonstrate the relevance of the smallcircle integrals in the dispersion representation for the vacuum polarization. Notice, that the finite small-circle contributions discussed in [10] come from the positronium poles. The latter are localized near the threshold and their contributions to the vacuum polarization are integrable.

The goal of our analysis is the algorithm of treating nonintegrable threshold singularities that emerge in perturbation theory; our algorithm allows one to obtain the analytic function as the solution of the integral equation and does not require the a priori knowledge of this analytic function.

In conclusion, in higher orders of the perturbation theory, nonintegrable threshold singularities given by powers of 1/v emerge. Our algorithm allows one to handle properly nonintegrable threshold singularities of two-point functions and leads to finite perturbative terms at each order; therefore, it may have a broad applicability. In particular, our algorithm opens the possibility of applying perturbative series to the investigation of ground-state hadrons within the machinery of QCD sum rules.

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Appendix A: Proof by induction for $F_{n+\frac{1}{2}}$ [odd powers $1/v^{2n+1}$]

We start with the integral over the cut

$$F_{n+\frac{1}{2}}(z,\epsilon) = \frac{z}{\pi} \int_{1+\epsilon}^{\infty} \mathrm{d}z' \frac{1}{(1-\frac{1}{z'})^{n+\frac{1}{2}}} \frac{1}{z'(z'-z)}.$$
 (A.1)

By virtue of (2.6), we come to a recursive relation

$$F_{n+\frac{1}{2}}(z,\epsilon) = \frac{z}{z-1} F_{n-\frac{1}{2}}(z,\epsilon) + \frac{1}{\pi} \frac{z}{1-z} \Delta F_{n+\frac{1}{2}}(\epsilon),$$
(A.2)

where

$$\Delta F_{n+\frac{1}{2}}(\epsilon) = \int_{1+\epsilon}^{\infty} dz' \frac{1}{(1-\frac{1}{z'})^{n+\frac{1}{2}}} \frac{1}{z'^2}$$
$$= \frac{\left(\frac{\epsilon}{1+\epsilon}\right)^{\frac{1}{2}-n} - 1}{n-\frac{1}{2}}.$$
(A.3)

This function is represented by an infinite sum of powers of $\sqrt{\epsilon}$, the most singular term being of the order $\epsilon^{-(n-\frac{1}{2})}$. Noteworthy, the ϵ -independent term in $\Delta F_{n+\frac{1}{2}}(\epsilon)$ is

$$-\frac{1}{(n-\frac{1}{2})}.$$
 (A.4)

We now conjecture that the ϵ -independent part of the recursive relation (A.2) yields the recursive relation for $F_{n+\frac{1}{2}}(z)$:

$$F_{n+\frac{1}{2}}(z) - F_{n+\frac{1}{2}}(0) = \frac{z}{z-1} \left\{ F_{n-\frac{1}{2}}(z) - F_{n-\frac{1}{2}}(0) \right\} + \frac{1}{\pi} \frac{z}{z-1} \frac{1}{(n-\frac{1}{2})}, \quad n \ge 1,$$
(A.5)

where the knowledge of the ϵ -independent term in $\Delta F_{n+\frac{1}{2}}(\epsilon)$ is used.

The analytic solution $F_{\frac{1}{2}}(z)$ is easily found:

$$F_{\frac{1}{2}}(z) = F_{\frac{1}{2}}(0) - \frac{2}{\pi} \sqrt{\frac{z}{z-1}} \log\left(\sqrt{-z} + \sqrt{1-z}\right).$$
(A.6)

Solving (A.5) iteratively, we find the conjectured solution $F_{n+\frac{1}{2}}(z)$. We still have to prove that this function indeed satisfies the integral equation (2.4). To this end, we need to calculate the small-circle integral of the function $F_{n+\frac{1}{2}}(z)$, see Eq. (2.4).

Using (2.6), the small-circle integral satisfies a recursive relation similar to (A.2):

$$I_{n+\frac{1}{2}}(z,\epsilon) = \frac{z}{z-1}I_{n-\frac{1}{2}}(z,\epsilon) + \frac{1}{\pi}\frac{z}{1-z}\Delta I_{n+\frac{1}{2}}(\epsilon),$$
(A.7)

where

$$\Delta I_{n+\frac{1}{2}}(\epsilon) = \frac{1}{2\pi i} \oint_{1+\epsilon} dz' F_{n+\frac{1}{2}}(z') \frac{1}{z'^2}.$$
 (A.8)

To calculate this small-circle integral, we expand the integrand near z = 1. Making use of the expansion

$$F_{n+\frac{1}{2}}(z) = (-1)^n \left(\frac{z}{1-z}\right)^{n+\frac{1}{2}} + O((z-1)^0)$$
(A.9)

and expanding also $1/z^2$ near z = 1, we find

$$\Delta F_{n+\frac{1}{2}}(\epsilon) + \Delta I_{n+\frac{1}{2}}(\epsilon) = -\frac{1}{(n-\frac{1}{2})},$$
(A.10)

yielding

$$F_{n+\frac{1}{2}}(z,\epsilon) + I_{n+\frac{1}{2}}(z,\epsilon)$$

= $\frac{z}{z-1} \left\{ F_{n-\frac{1}{2}}(z,\epsilon) + I_{n-\frac{1}{2}}(z,\epsilon) \right\}$
+ $\frac{z}{\pi(z-1)} \frac{1}{(n-\frac{1}{2})}.$ (A.11)

This equation coincides with Eq. (A.5) for

$$F_{n+\frac{1}{2}}(z) - F_{n+\frac{1}{2}}(0) = F_{n+\frac{1}{2}}(z,\epsilon) + I_{n+\frac{1}{2}}(z,\epsilon)$$
(A.12)

and verifies by induction that the sum $F_{n+\frac{1}{2}}(z,\epsilon) + I_{n+\frac{1}{2}}(z,\epsilon)$ does not depend on ϵ (since $F_{\frac{1}{2}}(z) - F_{\frac{1}{2}}(0)$ is ϵ -independent). We therefore prove our conjecture that the solution $F_{n+\frac{1}{2}}(z)$ is obtained from the dispersion integral over the cut, $F_{n+\frac{1}{2}}(z,\epsilon)$, by omitting *all* ϵ -dependent terms. Let us emphasize that the *only* role of the small-circle integral $I_{n+\frac{1}{2}}(z,\epsilon)$ is to cancel the ϵ -dependent terms in $F_{n+\frac{1}{2}}(z,\epsilon)$; it leaves no other traces.

Appendix B: Proof by induction for F_n [even powers $1/v^{2n}$]

This appendix gives the analysis for the case F_n which has some subtleties compared to the analysis for $F_{n+\frac{1}{2}}$ of Appendix A.

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We start with the integral over the cut

$$F_n(z,\epsilon) = \frac{z}{\pi} \int_{1+\epsilon}^{\infty} dz' \frac{1}{(1-\frac{1}{z'})^n} \frac{1}{z'(z'-z)}.$$
 (B.1)

By virtue of (2.6), one obtains a recursive relation

$$F_n(z,\epsilon) = \frac{z}{z-1}F_{n-1}(z,\epsilon) + \frac{1}{\pi}\frac{z}{1-z}\Delta F_n(\epsilon),$$
(B.2)

where

$$\Delta F_n(\epsilon) = \int_{1+\epsilon}^{\infty} dz' \frac{1}{(1-\frac{1}{z'})^n} \frac{1}{z'^2} = \frac{\left[(1+\frac{1}{\epsilon})^n - 1\right]\epsilon - 1}{(n-1)(1+\epsilon)}.$$
 (B.3)

For integer *n*, this function is represented by a finite sum of poles in $1/\epsilon$ of the increasing order; it contains no regular structures in ϵ .

Solving (B.2) for the ϵ -independent part and making use of the known expression for $F_1(z)$, we obtained $F_n(z)$ in the form [see (2.21)]:

$$F_n(z) = F_n(0) - \frac{1}{\pi} \left(\frac{z}{z-1}\right)^n \log(1-z).$$
 (B.4)

This is still our conjecture and we have to check that this function satisfies the integral equation (2.4).

Let us calculate the small-circle integral for $F_n(z)$ given by (B.4). Using again (2.6), we obtain also for the small-circle integral a recursive relation

$$I_n(z,\epsilon) = \frac{z}{z-1}I_{n-1}(z,\epsilon) + \frac{1}{\pi}\frac{z}{1-z}\Delta I_n(\epsilon),$$
(B.5)

where

$$\Delta I_n(\epsilon) = \frac{1}{2\pi i} \oint_{1+\epsilon} dz' \log(1-z') \frac{1}{(1-\frac{1}{z'})^n} \frac{1}{z'^2}$$
$$= -\frac{\left[(1+\frac{1}{\epsilon})^n - 1\right]\epsilon - 1}{(n-1)(1+\epsilon)} = -\Delta F_n(\epsilon), \tag{B.6}$$

Thus, $F_n(z, \epsilon) + I_n(z, \epsilon)$ satisfies the recursive relation

$$F_n(z,\epsilon) + I_n(z,\epsilon) = \frac{z}{z-1} \left\{ F_{n-1}(z,\epsilon) + I_{n-1}(z,\epsilon) \right\}.$$
 (B.7)

For n = 1 we find

$$F_{1}(z,\epsilon) = -\frac{1}{\pi} \frac{z}{z-1} \log(1-z) + \frac{1}{\pi} \frac{z}{z-1} \log(\epsilon) + \frac{z}{(z-1)^{2}} \epsilon + \frac{z}{2(z-1)^{3}} \epsilon^{2} + O(\epsilon^{3})$$
(B.8)

and

$$I_{1}(z,\epsilon) = -\frac{1}{\pi} \frac{z}{z-1} \log(\epsilon) -\frac{z}{(z-1)^{2}} \epsilon - \frac{z}{2(z-1)^{3}} \epsilon^{2} + O(\epsilon^{3}),$$
(B.9)

leading to

$$F_1(z,\epsilon) + I_1(z,\epsilon) = -\frac{1}{\pi} \frac{z}{z-1} \log(1-z).$$
 (B.10)

Let us emphasize again that not only singular in ϵ terms, but also powers of ϵ cancel in the sum, so $F_1(z, \epsilon) + I_1(z, \epsilon)$ does not depend on ϵ at all. As follows from (B.7), for any *n* the sum $F_n(z, \epsilon) + I_n(z, \epsilon)$ also does not depend on ϵ and has the form

$$F_n(z,\epsilon) + I_n(z,\epsilon) = -\frac{1}{\pi} \left(\frac{z}{z-1}\right)^n \log(1-z).$$
(B.11)

So our conjectured solution (B.4) indeed solves the integral equation (2.4).

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