ISSN 0025-6544, Mechanics of Solids, 2014, Vol. 49, No. 1, pp. 73–82. © Allerton Press, Inc., 2014.
Original Russian Text © V.I. Gorbachev, A.N. Emel'yanov, 2014, published in Izvestiya Akademii Nauk. Mekhanika Tverdogo Tela, 2014, No. 1, pp. 94–105.

Homogenization of the Equations of the Cosserat Theory of Elasticity of Inhomogeneous Bodies

V. I. Gorbachev* **and A. N. Emel'yanov****

Lomonosov Moscow State University, Leninskie Gory, Moscow, 119992 Russia

Received September 5, 2013

Abstract—The paper deals with the homogenization of a boundary value problem for an inhomogeneous body with Cosserat properties, which is referred to as the original problem. The homogenization process is understood as a method for representing the solution of the original problem in terms of the solution of precisely the same problem for a body with homogeneous properties. The problem for a body with homogeneous properties is called the accompanying problem, and the body itself, the accompanying homogeneous body. As a rule, a constructive homogenization procedure includes the following three stages: at the first stage, the properties of the inhomogeneous body are used to find the properties of the accompanying homogeneous body (efficient properties); at the second stage, the boundary value problem is solved for the accompanying body; at the third stage, the solution of the accompanying problem is used to find the solution of the original problem. This approach was implemented in mechanics of composite materials constructed of numerous representative elements. A significant contribution to the development of mechanics of composites is due to Rabotnov [1–3] and his students. Recently, the homogenization method has been widely used to solve problems for composites of regular structure by expanding the solution of the original problem in a power series in a small geometric parameter equal to the ratio of the characteristic dimension of the periodicity cell to the characteristic dimension of the entire body. The papers by Bakhvalov [4–6] and Pobedrya [7] were the first in the field. At present, there are numerous monographs partially or completely dealing with the method of a small geometric parameter [8–14].

Isolated problems for inhomogeneous bodies with nonperiodic dependence of their properties on the coordinates were considered by many authors. Most of such papers published before 1973 are collected in two vast bibliographic indices [15, 16]. General methods were considered, and many specific problems of the theory of elasticity of continuously inhomogeneous bodies were solved in Lomakin's papers and his monograph [17]. The theory of torsion of inhomogeneous anisotropic rods was considered in [18].

In 1991, in his Doctoral dissertation, one of the authors of this paper proposed a version of the homogenization method based on an integral formula representing the solution of the original static problem of inhomogeneous elasticity via the solution of the accompanying problem [19, 20]. An integral formula for the dynamic problem of elasticity was published somewhat later [21]. This integral formula was used to develop a constructive method for the homogenization of the dynamic problem of inhomogeneous elasticity, which can be used in the case of both periodic and nonperiodic inhomogeneity of the properties [22]. The integral formula in the case of the Cosserat theory of elasticity was published in [23]. The present paper briefly presents constructive methods for homogenizing the problems of the Cosserat theory of elasticity based on the integral formula.

DOI: 10.3103/S0025654414010099

Keywords: *elasticity, inhomogeneous body, Cosserat theory, composite, homogenization.*

1. STATEMENT OF THE ORIGINAL AND ACCOMPANYING PROBLEMS [24]

The Cosserat theory of elasticity, in addition to the stresses σ_{ij} and strains ε_{ij}, deals with the couple stress tensor μ_{ij} and the curvature tensor $\varkappa_{ij}$. All these tensors are nonsymmetric. The statement of the

*e-mail: vigorby@mail.ru
**e-mail: emlaldr@gmail.com

static problem of the Cosserat elasticity theory consists of the equilibrium equations

$$\sigma_{ji,j} + X_i = 0, \quad \mu_{ji,j} + \epsilon_{ijk}\sigma_{jk} + Y_i = 0, \tag{1.1}$$

the constitutive relations

$$\sigma_{ji} = C_{ijmn}\varepsilon_{nm} + B_{ijmn}\varkappa_{nm}, \quad \mu_{ji} = B_{ijmn}\varepsilon_{nm} + D_{ijmn}\varkappa_{nm}, \tag{1.2}$$

the Cauchy type relations

$$\varepsilon_{nm} = u_{m,n} + \epsilon_{mns}\omega_s, \quad \varkappa = \omega_{m,n}, \tag{1.3}$$

expressing the strains and curvatures in terms of components of the displacement vector u_i and the rotation vector ω_i, and the boundary conditions

$$u_i\big|_\Sigma = u_i^0, \quad \omega_i\big|_\Sigma = \omega_i^0. \tag{1.4}$$

The coefficients C_{ijmn}, D_{ijmn}, and B_{ijmn} are components of tensors of rank four. They are symmetric with respect to the first and second pairs of indices but are not symmetric with respect to the indices in pairs. The physical dimension of these tensors is different: $[\underset{\sim}{\mu}] = [\ell]^1[\underset{\sim}{\sigma}]$, $[\underset{\sim}{C}] = [\ell]^0[\underset{\sim}{\sigma}]$, $[\underset{\sim}{B}] = [\ell]^1[\underset{\sim}{\sigma}]$, and $[\underset{\sim}{D}] = [\ell]^2[\underset{\sim}{\sigma}]$. Here ℓ is the length referred to the structure of the material (structure parameter), for example, the characteristic dimension of inhomogeneity, the characteristic dimension of the composite representative element, or the characteristic size of the periodicity cell for a composite with a regular structure. The square brackets containing a symbol denote the dimension of the variable marked by this symbol.

The accompanying problem is a problem similar to the original problem for a body of the same shape and with the same initial data but with different material characteristics C^0_{ijkl}, D^0_{ijkl}, and B^0_{ijkl}. Let v_i, e_{ij}, and τ_{ij} denote the displacements, strains, and stresses, and let ψ_i, π_{ij}, and ν_{ij} denote the angles of rotation, couple strains, and couple stresses in the corresponding problem. The statement of the accompanying problem is given by the formulas

$$\tau_{ji,j} + X_i = 0, \quad \nu_{ji,j} + \epsilon_{ijk}\tau_{jk} + Y_i = 0, \tag{1.5}$$

$$\tau_{ji} = C^0_{ijkl}e_{lk} + B^0_{ijkl}\pi_{lk}, \quad \nu_{ji} = B^0_{ijkl}e_{lk} + D^0_{ijkl}\pi_{lk}, \tag{1.6}$$

$$e_{lk} = v_{k,l} + \epsilon_{kls}\psi_s, \quad \pi_{lk} = \psi_{k,l}, \tag{1.7}$$

$$v_i\big|_\Sigma = u_i^0, \quad \psi_i\big|_\Sigma = \omega_i^0. \tag{1.8}$$

The equations of the accompanying problem can be reduced to the system of equations

$$\begin{aligned} &C^0_{ijkl}e_{lk,j} + B^0_{ijkl}\pi_{lk,j} + X_i = 0, \\ &B^0_{ijkl}e_{lk,j} + D^0_{ijkl}\pi_{lk,j} + \epsilon_{ijr}(C^0_{rjkl}e_{lk} + B^0_{rjkl}\pi_{lk}) + Y_i = 0. \end{aligned} \tag{1.9}$$

2. INTEGRAL FORMULAS IN THE STATIC PROBLEM OF THE COSSERAT THEORY OF ELASTICITY FOR INHOMOGENEOUS BODIES

The following integral formulas relating the solutions of boundary-value problems of the same type for inhomogeneous and homogeneous elastic bodies of the same shape and with the same initial data were derived in [23],

$$\begin{aligned} u_i(x) = v_i(x) &+ \int\limits_V \big\{\overset{1(i)}{\varepsilon}_{kl}(x,\xi)[C^0_{klpq} - C_{klpq}(\xi)] + \overset{1(i)}{\kappa}_{kl}(x,\xi)[B^0_{klpq} - B_{klpq}(\xi)]\big\}e_{pq}(\xi)\,dV_\xi \\ &+ \int\limits_V \big\{\overset{1(i)}{\varepsilon}_{kl}(x,\xi)[B^0_{klpq} - B_{klpq}(\xi)] + \overset{1(i)}{\kappa}_{kl}(x,\xi)[D^0_{klpq} - D_{klpq}(\xi)]\big\}\pi_{pq}(\xi)\,dV_\xi, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \omega_i(x) = \psi_i(x) &+ \int\limits_V \big\{\overset{2(i)}{\varepsilon}_{kl}(x,\xi)[C^0_{klpq} - C_{klpq}(\xi)] + \overset{2(i)}{\kappa}_{kl}(x,\xi)[B^0_{klpq} - B_{klpq}(\xi)]\big\}e_{pq}(\xi)\,dV_\xi \\ &+ \int\limits_V \big\{\overset{2(i)}{\varepsilon}_{kl}(x,\xi)[B^0_{klpq} - B_{klpq}(\xi)] + \overset{2(i)}{\kappa}_{kl}(x,\xi)[D^0_{klpq} - D_{klpq}(\xi)]\big\}\pi_{pq}(\xi)\,dV_\xi, \end{aligned} \tag{2.2}$$

$$\overset{a(i)}{\varepsilon}_{kl}(x,\xi) = \overset{a(i)}{u}_{l,k}(x,\xi) + \epsilon_{lks}\overset{a(i)}{\omega}_{s}(x,\xi), \quad \overset{a(i)}{\kappa}_{kl}(x,\xi) = \overset{a(i)}{\omega}_{l,k}(x,\xi), \quad a = 1,2,$$

where $\overset{a(i)}{u}_{l,k}(x,\xi)$ and $\overset{a(i)}{\omega}_{l,k}(x,\xi)$ are components of the displacement tensor and Green's rotation tensor. In the static problem of the Cosserat theory of elasticity, Green's tensors can be introduced in two different ways. In the first case, at a point ξ of the body, the unit lumped force directed along the x_k-axis is prescribed, and this force causes a displacement $\overset{1(k)}{u}_{i}(x,\xi)$ and a rotation $\overset{1(k)}{\omega}_{i}(x,\xi)$. In the second case, at a point ξ of the body, a unit lumped moment directed along the x_k-axis is prescribed. Then the displacement $\overset{2(k)}{u}_{i}(x,\xi)$ and the rotation $\overset{1(k)}{\omega}_{i}(x,\xi)$ arise at a point x of the body. In both cases, the boundary conditions are assumed to be zero.

3. REPRESENTATION IN THE FORM OF SERIES

We assume that the strains and curvatures in the accompanying problem are smooth functions of the coordinates x_i. Then, in a neighborhood of any point $\xi \subset V$, they can be represented in terms of the values at the point $x \subset V$ as the Taylor series

$$e_{kl}(\xi) = \sum_{q=0}^{\infty} \Pi_{i_1\ldots i_q}(\xi,x) e_{kl,i_1\ldots i_q}(x), \quad \pi_{kl}(\xi) = \sum_{q=0}^{\infty} \Pi_{i_1\ldots i_q}(\xi,x) \pi_{kl,i_1\ldots i_q}(x), \tag{3.1}$$

$$\Pi_{i_1\ldots i_q}(\xi,x) \equiv 1q!(\xi_{i_1} - x_{i_1})\cdots(\xi_{i_q} - x_{i_q}). \tag{3.2}$$

By substituting the expressions (3.1) into formulas (2.1) and (2.2), we obtain the following representation for the solution of the original problem of the Cosserat theory of elasticity in the form of series in all possible derivatives of the strains and curvatures in the accompanying problem:

$$u_i(x) = v_i(x) + \sum_{q=0}^{\infty} [N_{imn(q)}(x)\partial_q e_{nm}(x) + U_{imn(q)}(x)\partial_q \pi_{nm}(x)], \tag{3.3}$$

$$\omega_i(x) = \psi_i(x) + \sum_{q=0}^{\infty} [V_{imn(q)}(x)\partial_q e_{nm}(x) + M_{imn(q)}(x)\partial_q \pi_{nm}(x)]. \tag{3.4}$$

Here we write out the formulas in an abbreviated form such that, for example, $\delta_{ji_q} N_{imn(q-1)}\partial_q e_{nm} \equiv \delta_{ji_q} N_{imni_1\ldots i_{q-1}} e_{nm,i_1\ldots i_q}$.

The coefficients in the series $U_{imn(q)}$, $U_{imn(q)}$, $V_{imn(q)}$, and $M_{imn(q)}$ are continuous functions of coordinates vanishing on the boundary of the body: $(N,U,M,V)_{imn(q)}\big|_{\Sigma} = 0$. They are weighted moments of strains and Green's curvatures of the initial problem [25]. These functions are identically zero for a homogeneous material and different from zero in the case of an inhomogeneous material. The form of $(N,U,M,V)_{imn(q)}$-functions is determined by the functional dependence on the coordinates of the physical and mechanical characteristics of the material, and hence it is meaningful to call them the *structure functions*. The functions $(N,U,M,V)_{imn(q)}$ form *structure tensors* of rank $q+3$.

The structure functions have the following physical dimension: $[N_{imn(q)}] = [\ell]^{q+1}$, $[U_{imn(q)}] = [\ell]^{q+2}$, $[V_{imn(q)}] = [\ell]^{q}$, and $[M_{imn(q)}] = [\ell]^{q+1}$. The series for the strains and curvatures become

$$\begin{aligned} \varepsilon_{ji} = e_{ji} + \sum_{q=0}^{\infty} \big\{ & [N_{imn(q),j} + \delta_{ji_q} N_{imn(q-1)} + \epsilon_{ijs} V_{smn(q)}]\partial_q e_{nm} \\ & + [U_{imn(q),j} + \delta_{ji_q} U_{imn(q-1)} + \epsilon_{ijs} M_{smn(q)}]\partial_q \pi_{nm} \big\}, \end{aligned} \tag{3.5}$$

$$\varkappa_{ji} = \pi_{ji} + \sum_{q=0}^{\infty} \big\{ [V_{imn(q),j} + \delta_{ji_q} V_{imn(q-1)}]\partial_q e_{nm} + [M_{imn(q),j} + \delta_{ji_q} M_{imn(q-1)}]\partial_q \pi_{nm} \big\}. \tag{3.6}$$

After this, we write out the expressions for the force and couple stresses,

$$\sigma_{ji} = \sum_{q=0}^{\infty} [\tilde{C}_{ijmn(q)}\partial_q e_{nm} + \tilde{B}_{ijmn(q)}\partial_q \pi_{nm}], \tag{3.7}$$

$$\mu_{ji} = \sum_{q=0}^{\infty} [\tilde{\tilde{B}}_{ijmn(q)} \partial_q e_{nm} + \tilde{\tilde{D}}_{ijmn(q)} \partial_q \pi_{nm}]. \tag{3.8}$$

The coefficients in the series (3.7) and (3.8) for the stresses have the following dimension:

$$[\tilde{C}_{ijmn(q)}] = [l]^q[\sigma], \quad [\tilde{B}_{ijmn(q)}] = [\tilde{\tilde{B}}_{ijmn(q)}] = [l]^{q+1}[\sigma], \quad [\tilde{D}_{ijmn(q)}] = [l]^{q+2}[\sigma].$$

The coefficients $\tilde{C}_{ijmn(q)}$, $\tilde{B}_{ijmn(q)}$, $\tilde{\tilde{B}}_{ijmn(q)}$, and $\tilde{D}_{ijmn(q)}$ can be expressed via the $(N, U, M, V)_{imn(q)}$-functions (the index (0) is omitted in the case $q = 0$) as follows:

$$\tilde{C}_{ijmn(0)} \equiv \tilde{C}_{ijmn} = C_{ijmn} + C_{ijkl}(N_{kmn,l} + \epsilon_{kls} V_{smn}) + B_{ijkl} V_{kmn,l}, \tag{3.9}$$

$$\tilde{B}_{ijmn(0)} \equiv \tilde{B}_{ijmn} = B_{ijmn} + C_{ijkl}(U_{kmn,l} + \epsilon_{kls} M_{smn}) + B_{ijkl} M_{kmn,l}, \tag{3.10}$$

$$\tilde{\tilde{B}}_{ijmn(0)} \equiv \tilde{\tilde{B}}_{ijmn} = B_{ijmn} + B_{ijkl}(N_{kmn,l} + \epsilon_{kls} V_{smn}) + D_{ijkl} V_{kmn,l}, \tag{3.11}$$

$$\tilde{D}_{ijmn(0)} \equiv \tilde{D}_{ijmn} = D_{ijmn} + B_{ijkl}(U_{kmn,l} + \epsilon_{kls} M_{smn}) + D_{ijkl} M_{kmn,l}, \tag{3.12}$$

and for $q \geq 1$ as follows:

$$\tilde{C}_{ijmn(q)} = C_{ijkl}[N_{kmn(q),l} + \delta_{li_q} N_{kmn(q-1)} + \epsilon_{kls} V_{smn(q)}] + B_{ijkl}[V_{kmn(q),l} + \delta_{li_q} V_{kmn(q-1)}], \tag{3.13}$$

$$\tilde{B}_{ijmn(q)} = C_{ijkl}[U_{kmn(q),l} + \delta_{li_q} U_{kmn(q-1)} + \epsilon_{kls} M_{smn(q)}] + B_{ijkl}[M_{kmn(q),l} + \delta_{li_q} M_{kmn(q-1)}], \tag{3.14}$$

$$\tilde{\tilde{B}}_{ijmn(q)} = B_{ijkl}[N_{kmn(q),l} + \delta_{li_q} N_{kmn(q-1)} + \epsilon_{kls} V_{smn(q)}] + D_{ijkl}[V_{kmn(q),l} + \delta_{li_q} V_{kmn(q-1)}], \tag{3.15}$$

$$\tilde{D}_{ijmn(q)} = B_{ijkl}[U_{kmn(q),l} + \delta_{li_q} U_{kmn(q-1)} + \epsilon_{kls} M_{smn(q)}] + D_{ijkl}[M_{kmn(q),l} + \delta_{li_q} M_{kmn(q-1)}]. \tag{3.16}$$

We substitute the series (3.7) and (3.8) into the equations of the original problem and obtain

$$\begin{gathered}
\sum_{q=0}^{\infty} \big[(\tilde{\tilde{C}}_{ijmn(q),j} + \tilde{\tilde{C}}_{ii_jmn(q-1)}) \partial_q e_{nm} + (\tilde{\tilde{B}}_{ijmn(q),j} + \tilde{\tilde{B}}_{ii_jmn(q-1)}) \partial_q \pi_{nm}\big] + X_i = 0, \\
\sum_{q=0}^{\infty} \big[(\tilde{\tilde{B}}_{ijmn(q),j} + \tilde{\tilde{B}}_{ii_jmn(q-1)} + \varepsilon_{ijr} \tilde{C}_{rjmn(q)}) \partial_q e_{nm} \\
+ (\tilde{\tilde{D}}_{ijmn(q),j} + \tilde{\tilde{D}}_{ii_jmn(q-1)} + \epsilon_{ijr} \tilde{B}_{rjimn(q)}) \partial_q \pi_{nm}\big] + Y_i = 0.
\end{gathered} \tag{3.17}$$

Then we rewrite Eqs. (1.9) of the accompanying problem as follows:

$$\begin{gathered}
C^0_{ii_1mn} e_{nm,i_1} + B^0_{ii_1mn} \pi_{nm,i_1} + X_i = 0, \\
B^0_{ii_1mn} e_{nm,i_1} + D^0_{ii_1mn} \pi_{nm,i_1} + \epsilon_{ijr}(C^0_{rjmn} e_{nm} + B^0_{rjmn} \pi_{nm}) + Y_i = 0.
\end{gathered} \tag{3.18}$$

By comparing Eqs. (3.17) and (3.18), we see that the coefficients of the derivatives of e_{ij} and π_{ij} in Eqs. (3.17) should be as follows:

the first group of equalities:

$$q = 0 \quad \begin{cases} \tilde{C}_{ijmn(0),j} = 0, \\ \tilde{\tilde{B}}_{ijmn(0),j} + \epsilon_{ijr} \tilde{C}_{rjmn(0)} = \varepsilon_{ijr} C^0_{rjmn}, \end{cases} \tag{3.19}$$

$$q = 1 \quad \begin{cases} \tilde{C}_{ijmn(1),j} + \tilde{C}_{ii_1mn(0)} = C^0_{ii_1mn}, \\ \tilde{\tilde{B}}_{ijmn(1),j} + \tilde{\tilde{B}}_{ii_1mn(0)} + \epsilon_{ijr} \tilde{C}_{rjmn(1)} = B^0_{ii_1mn}, \end{cases} \tag{3.20}$$

$$q \geq 2 \quad \begin{cases} \tilde{C}_{ijmn(q),j} + \tilde{C}_{ii_qmn(q-1)} = 0, \\ \tilde{\tilde{B}}_{ijmn(q),j} + \tilde{\tilde{B}}_{ii_qmn(q-1)} + \epsilon_{ijr} \tilde{C}_{rjmn(q)} = 0; \end{cases} \tag{3.21}$$

the second group of equalities:

$$q = 0 \quad \begin{cases} \tilde{B}_{ijmn(0),j} = 0, \\ \tilde{\tilde{D}}_{ijmn(0),j} + \epsilon_{ijr} \tilde{B}_{rjmn(0)} = \varepsilon_{ijr} B^0_{rjmn}, \end{cases} \tag{3.22}$$

$$q = 1 \quad \begin{cases} \tilde{B}_{ijmn(1),j} + \tilde{B}_{ii_1mn(0)} = B^0_{ii_1mn}, \\ \tilde{\tilde{D}}_{ijmn(1),j} + \tilde{\tilde{D}}_{ii_1mn(0)} + \epsilon_{ijr}\tilde{B}_{rjmn(1)} = D^0_{ii_1mn}, \end{cases} \tag{3.23}$$

$$q \geq 2 \quad \begin{cases} \tilde{B}_{ijmn(q),j} + \tilde{B}_{ii_qmn(q-1)} = 0, \\ \tilde{\tilde{D}}_{ijmn(q),j} + \tilde{\tilde{D}}_{ii_qmn(q-1)} + \epsilon_{ijr}\tilde{B}_{rjmn(q)} = 0. \end{cases} \tag{3.24}$$

By substituting (3.9), (3.11) and (3.13), (3.15) into the first group of equalities (3.19)–(3.21), we obtain the following system of recursion equations for the functions $N_{kmn(q)}$ and $V_{kmn(q)}$. The recursion in the first group starts from the equations

$$\begin{aligned} &[C_{ijmn} + C_{ijkl}(N_{kmn,l} + \epsilon_{kls}V_{smn}) + B_{ijkl}V_{kmn,l}]_j = 0, \\ &[B_{ijmn} + B_{ijkl}(N_{kmn,l} + \epsilon_{kls}V_{smn}) + D_{ijkl}V_{kmn,l}]_j = \epsilon_{ijr}(C^0_{rjmn} - \tilde{C}_{rjmn}). \end{aligned} \tag{3.25}$$

The functions $M_{kmn(q)}$ and $U_{kmn(q)}$ are determined from the second group of equations. The recursion in the first group starts from the equations

$$\begin{aligned} &[B_{ijmn} + C_{ijkl}(U_{kmn,l} + \epsilon_{kls}M_{smn}) + B_{ijkl}M_{kmn,l}]_j = 0, \\ &[B_{ijmn} + B_{ijkl}(N_{kmn,l} + \epsilon_{kls}V_{smn}) + D_{ijkl}V_{kmn,l}]_j = \epsilon_{ijr}(C^0_{rjmn} - \tilde{C}_{rjmn}). \end{aligned} \tag{3.26}$$

The unique solution of Eqs. (3.25) and (3.26) and of all subsequent equations is determined by the condition that all $(N, U, V, M)_{imn(q)}$ functions are zero on the boundary Σ of the domain V occupied by the body.

The solution of the original problems in the form of series contains the coefficients C^0_{ijkl}, D^0_{ijkl}, and B^0_{ijkl}, which represent the accompanying homogeneous body. In principle, these can be any physically admissible quantities. Obviously, they do not affect the exact solution of the original problem. But the choice of the properties of the accompanying body significantly affects the rate of convergence of the series to the exact solution. In [25], it is proposed to choose them in the form

$$C^0_{ijmn} = \langle \tilde{C}_{ijmn} \rangle_V = \langle C_{ijmn} + C_{ijkl}(N_{kmn,l} + \epsilon_{kls}V_{smn}) + B_{ijkl}V_{kmn,l} \rangle_V, \tag{3.27}$$

$$D^0_{ijmn} = \langle \tilde{D}_{ijmn} \rangle_V = \langle D_{ijmn} + B_{ijkl}(U_{kmn,l} + \epsilon_{kls}M_{smn}) + D_{ijkl}M_{kmn,l} \rangle_V, \tag{3.28}$$

$$\begin{aligned} B^0_{ijmn} &= \langle \tilde{\tilde{B}}_{ijmn} \rangle_V = \langle B_{ijmn} + B_{ijkl}(N_{kmn,l} + \epsilon_{kls}V_{smn}) + D_{ijkl}V_{kmn,l} \rangle_V \\ &= \langle \tilde{B}_{ijmn} \rangle_V = \langle B_{ijmn} + C_{ijkl}(U_{kmn,l} + \epsilon_{kls}M_{smn}) + B_{ijkl}M_{kmn,l} \rangle_V. \end{aligned} \tag{3.29}$$

Let us show that these are precisely the effective coefficients in the sense of the definition given below.

4. EFFECTIVE CHARACTERISTICS IN THE COSSERAT THEORY OF ELASTICITY

The effective coefficients of any inhomogeneous Cosserat material composed of identical representative volumes are coefficients that permit relating the force and couple stresses $\langle \sigma_{ij} \rangle$ and $\langle \mu_{ij} \rangle$ averaged over any representative volume to the strains $\langle \varepsilon_{ij} \rangle$ and curvatures $\langle \varkappa_{ij} \rangle$ averaged over the same representative volume. This definition of effective properties of a Cosserat elastic body is a generalization of the definition given in [26] to the case of a body composed of representative volumes of a material with Cosserat properties.

For simplicity, first consider a periodically inhomogeneous material with periodicity cells Ω of cubic shape with edge ℓ. In this case, the cube Ω is a representative volume, and its edge ℓ is a structure parameter. In a periodically inhomogeneous body, the material tensors C, B, and D are one-periodic functions of the local variables $0 \leq \zeta_i = \{x_i/\ell\} = x_i/\ell - [x_i/\ell] \leq 1$. Here the braces denote the fractional part of a number, and the square brackets denote the integer part of a number. The variables ζ_i are also called the fast variables [6], and functions of the fast variables are called rapidly oscillating functions of the global coordinates x_i [17].

We introduce new structure functions and structure tensors by using the replacements $N_{imn(q)} \to \ell^{q+1}N_{imn(q)}$, $U_{imn(q)} \to \ell^{q+2}U_{imn(q)}$, $V_{imn(q)} \to \ell^q V_{imn(q)}$, $M_{imn(q)} \to \ell^{q+1}M_{imn(q)}$, $B_{ijmn} \to \ell B_{ijmn}$, and $D_{ijmn} \to \ell^2 D_{ijmn}$.

The new functions depend on the local variables ζ_i. Moreover, $(N,U,V,M)_{imn(q)}$ are dimensionless functions, and the new material coefficients B_{ijmn} and D_{ijmn} are of the dimension of stresses. The functions of local coordinates ζ_i are differentiated with respect to the global coordinates x_i according to the rule

$$f_{,i}(\zeta) \equiv \frac{\partial f(\zeta)}{\partial x_i} = \frac{\partial f}{\partial \zeta_k}\frac{\partial \zeta_k}{\partial x_i} = \frac{1}{\ell}\frac{\partial f}{\partial \zeta_i k} \equiv 1\ell f_{|i}.$$

After this, the structure parameter ℓ occurs in most of the formulas in the preceding sections; for example, the constitutive relations (1.2) of the original problem become

$$\sigma_{ji} = C_{ijmn}(\zeta)\varepsilon_{nm} + \ell B_{ijmn}(\zeta)\varkappa_{nm}, \quad \mu_{ji} = \ell B_{ijmn}(\zeta)\varepsilon_{nm} + \ell^2 D_{ijmn}(\zeta)\varkappa_{nm}, \tag{4.1}$$

and the effective constitutive relations of the form $(\langle\sigma\rangle, \langle\mu\rangle) \sim (\langle\varepsilon\rangle, \langle\varkappa\rangle)$ will be written as

$$\langle\sigma_{ji}\rangle_\Omega = C^{\text{eff}}_{ijmn}\langle\varepsilon_{nm}\rangle_\Omega + \ell B^{\text{eff}}_{ijmn}\langle\varkappa_{nm}\rangle_\Omega, \quad \langle\mu_{ji}\rangle_\Omega = \ell B^{\text{eff}}_{ijmn}\langle\varepsilon_{nm}\rangle_\Omega + \ell^2 D^{\text{eff}}_{ijmn}\langle\varkappa_{nm}\rangle_\Omega. \tag{4.2}$$

According to the new structure functions, the series (3.3)–(3.8) become

$$u_i(x) = v_i(x) + \sum_{q=0}^{\infty}[\ell^{q+1} N_{imn(q)}(\zeta)\partial_q e_{nm}(x) + \ell^{q+2} U_{imn(q)}(\zeta)\partial_q \pi_{nm}(x)], \tag{4.3}$$

$$\omega_i(x) = \psi_i(x) + \sum_{q=0}^{\infty}[\ell^{q} V_{imn(q)}(\zeta)\partial_q e_{nm}(x) + \ell^{q+1} M_{imn(q)}(\zeta)\partial_q \pi_{nm}(x)], \tag{4.4}$$

$$\begin{aligned}\varepsilon_{ji} = {} & (\delta_{jn}\delta_{im} + N_{imn|j} + \epsilon_{ijs}V_{smn})e_{nm} + \ell(U_{imn|j} + \varepsilon_{ijs}M_{smn})\pi_{mn} \\ & + \sum_{q=1}^{\infty}[\ell^q(N_{imn(q)|j} + N_{imn(q-1)}\delta_{ji_q} + \epsilon_{ijs}V_{smn(q)})\partial_q e_{nm} \\ & + \ell^{q+1}(U_{imn(q)|j} + U_{imn(q-1)}\delta_{ji_q} + \epsilon_{ijs}M_{smn(q)})\partial_q \pi_{nm}],\end{aligned} \tag{4.5}$$

$$\begin{aligned}\varkappa_{ji} = {} & \frac{1}{\ell}V_{imn|j}e_{nm} + (\delta_{jn}\delta_{im} + M_{imn|j})\pi_{nm} \\ & + \sum_{q=1}^{\infty}[\ell^{q-1}(V_{imn(q)|j} + V_{imn(q-1)}\delta_{ji_q})\partial_q e_{nm} + \ell^q(M_{imn(q)|j} + M_{imn(q-1)}\delta_{ji_q})\partial_q \pi_{nm}],\end{aligned} \tag{4.6}$$

$$\sigma_{ji} = \sum_{q=0}^{\infty}[\ell^q \tilde{C}_{ijmn(q)}\partial_q e_{nm} + \ell^{q+1}\tilde{B}_{ijmn(q)}\partial_q \pi_{nm}], \tag{4.7}$$

$$\mu_{ji} = \sum_{q=0}^{\infty}[\ell^{q+1}\tilde{\tilde{B}}_{ijmn(q)}\partial_q e_{nm} + \ell^{q+2}\tilde{D}_{ijmn(q)}\partial_q \pi_{nm}]. \tag{4.8}$$

The recursion equations for the new $(N,U,V,M)_{imn(q)}$ functions retain the form (3.19)–(3.26) except that the derivative with respect to the global coordinate, which is denoted by an index after the comma in (3.19)–(3.26), is replaced by the derivative with respect to the local variable, which is denoted by an index after the vertical bar. In the general case, to determine the $(N,U,V,M)_{imn(q)}$ functions, one should solve the boundary value problems for Eqs. (3.19)–(3.26) with homogeneous conditions on the boundary of the entire body, The coefficients in these equations are periodic functions of the local variables. At the distance of the order of the characteristic dimension of the periodicity cell from the body boundary, the desired functions also tend to periodic functions [27] that are continuous and periodic solutions of Eqs. (3.19)–(3.26) in the cube Ω. This fact is pointed out in [28–30] and is numerically confirmed in the dissertation [31]. The periodic solutions of Eqs. (3.19)–(3.26) in the cube are determined up to constants [6], which can be found from the normalization condition $\langle(N,U,V,M)_{imn(q)}\rangle_\Omega = 0$.

Let L be the characteristic dimension of the entire body, and let $\ell/L \ll 1$; i.e., the body is composed of a large number of cells in all directions. In this case, the smooth functions $\partial_q e_{nm}(x)$ and $\partial_q \pi_{nm}(x)$ do not practically change in any ℓ-cube; i.e., they behave as constants after averaging over the ℓ-cube,

$$\langle f_{(q)}(\zeta)\partial_q e_{nm}(x)\rangle_\Omega \approx \langle f_{(q)}(\zeta)\rangle_\Omega \partial_q e_{nm}(x), \quad \langle f_{(q)}(\zeta)\partial_q \pi_{nm}(x)\rangle_\Omega \approx \langle f_{(q)}(\zeta)\rangle_\Omega \partial_q \pi_{nm}(x).$$

It follows from the above that the averaging over any inner cell of the expressions (4.3), (4.4) for the components of the displacements and rotation vectors and of the expressions (4.5), (4.8) for the strains and the curvature gives

$$\langle u_i\rangle_\Omega \to v_i, \quad \langle \omega_i\rangle_\Omega \to \psi_i \quad \text{as } \alpha \equiv \ell/L \to 0, \tag{4.9}$$

$$\langle \varepsilon_{ij}\rangle_\Omega \to e_{ij}, \quad \langle \varkappa_{ij}\rangle_\Omega \to \pi_{ij} \quad \text{as } \alpha \equiv \ell/L \to 0. \tag{4.10}$$

The expressions (4.7) and (4.8) for the force and couple stresses averaged over the periodicity cell can be represented as

$$\langle \sigma_{ji}\rangle_\Omega = \langle \tilde{C}_{ijmn}\rangle_\Omega e_{mn} + \ell\langle \tilde{B}_{ijmn}\rangle_\Omega \pi_{mn} + O(\alpha) \xrightarrow{\alpha\to 0} \langle \tilde{C}_{ijmn}\rangle_\Omega \langle \varepsilon_{ji}\rangle_\Omega + \ell\langle \tilde{B}_{ijmn}\rangle_\Omega \langle \varkappa_{ij}\rangle_\Omega, \tag{4.11}$$

$$\langle \mu_{ji}\rangle_\Omega = \ell\langle \tilde{\tilde{B}}_{ijmn}\rangle_\Omega e_{mn} + \ell^2\langle \tilde{D}_{ijmn}\rangle_\Omega \pi_{mn} + O(\alpha^2) \xrightarrow{\alpha\to 0} \ell\langle \tilde{\tilde{B}}_{ijmn}\rangle_\Omega \langle \varepsilon_{ij}\rangle_\Omega + \ell^2\langle \tilde{D}_{ijmn}\rangle_\Omega \langle \varkappa_{ij}\rangle_\Omega. \tag{4.12}$$

These formulas and definition (4.2) give the following expressions for the effective characteristics of a composite with Cosserat properties:

$$C^{\text{eff}}_{ijmn} = \langle \tilde{C}_{ijmn}\rangle_\Omega = \langle C_{ijmn} + C_{ijkl}(N_{kmn|l} + \epsilon_{kls}V_{smn}) + B_{ijkl}V_{kmn|l}\rangle_\Omega, \tag{4.13}$$

$$D^{\text{eff}}_{ijmn} = \langle \tilde{D}_{ijmn}\rangle_\Omega = \langle D_{ijmn} + B_{ijkl}(U_{kmn|l} + \epsilon_{kls}M_{smn}) + D_{ijkl}M_{kmn|l}\rangle_\Omega, \tag{4.14}$$

$$\begin{aligned} B^{\text{eff}}_{ijmn} &= \langle \tilde{B}_{ijmn}\rangle_\Omega = \langle \tilde{\tilde{B}}_{ijmn}\rangle_\Omega = \langle B_{ijmn} + B_{ijkl}(N_{kmn|l} + \epsilon_{kls}V_{smn}) + D_{ijkl}V_{kmn|l}\rangle_\Omega \\ &= \langle B_{ijmn} + C_{ijkl}(U_{kmn|l} + \epsilon_{kls}M_{smn}) + B_{ijkl}M_{kmn|l}\rangle_\Omega. \end{aligned} \tag{4.15}$$

Thus, to obtain the effective characteristics of regular composites, one should average the functions $\tilde{C}_{ijkl}(\zeta)$, $\tilde{B}_{ijkl}(\zeta)$, and $\tilde{D}_{ijkl}(\zeta)$ over the periodicity cell. The functions $(N,U,V,M)_{imn(q)}$ in formulas (4.13)–(4.15) are determined from the solution of the coupled systems (3.15) and (3.26) in the periodicity cell. The unique solution of these systems is chosen from the periodicity conditions

$$N_{kmn}(\zeta_1,\zeta_2,\zeta_3)\big|_{\zeta_i=0} = N_{kmn}(\zeta_1,\zeta_2,\zeta_3)\big|_{\zeta_i=1} \quad (i=1,2,3), \tag{4.16}$$

$$V_{kmn}(\zeta_1,\zeta_2,\zeta_3)\big|_{\zeta_i=0} = V_{kmn}(\zeta_1,\zeta_2,\zeta_3)\big|_{\zeta_i=1} \quad (i=1,2,3), \tag{4.17}$$

$$U_{kmn}(\zeta_1,\zeta_2,\zeta_3)\big|_{\zeta_i=0} = U_{kmn}(\zeta_1,\zeta_2,\zeta_3)\big|_{\zeta_i=1} \quad (i=1,2,3), \tag{4.18}$$

$$M_{kmn}(\zeta_1,\zeta_2,\zeta_3)\big|_{\zeta_i=0} = M_{kmn}(\zeta_1,\zeta_2,\zeta_3)\big|_{\zeta_i=1} \quad (i=1,2,3), \tag{4.19}$$

and the normalization conditions

$$\langle N_{kmn}\rangle = 0, \quad \langle V_{kmn}\rangle = 0, \quad \langle U_{kmn}\rangle = 0, \quad \langle M_{kmn}\rangle = 0. \tag{4.20}$$

In the general case of inhomogeneity, i.e., in the case where the elasticity coefficients are arbitrary integrable functions of the global coordinates x, the effective characteristics are determined by the same formulas (4.13)–(4.15), but the averaging in these formulas is performed over the entire body by formulas (3.27)–(3.29). The functions $(N,U,V,M)_{imn(q)}$ depend on x and are determined by solving the same equations (3.25) and (3.26) in the entire inhomogeneous body for zero values of the desired functions on the boundary of the body. In the specific case where the elasticity coefficients are periodic, the functions $(N,U,V,M)_{imn(q)}(x)$ significantly differ from periodic functions only in the boundary layer, whose thickness is equal to several characteristic dimensions of the periodicity cell and tends to zero with further refinement of the structure. Thus, the elastic characteristics determined by various formulas differ only by quantities of order $O(\alpha)$.

5. CASE OF A LAYER INFINITE IN THE HORIZONTAL PROJECTION AND INHOMOGENEOUS ACROSS THE THICKNESS

Let L denote the plate thickness. Assume that the x_3-axis is perpendicular to the face surfaces of the layer and the lower face surface corresponds to the value $x_3 = 0$. In this case, the coefficients C_{ijkl}, D_{ijkl}, and B_{ijkl} are functions of the coordinate x_3. We assume that the desired functions $N_{kmn(q)}$, $U_{kmn(q)}$, $V_{kmn(q)}$, and $M_{kmn(q)}$ depend only on x_3 as well. Because of (3.27)–(3.19), Eqs. (3.25) and (3.26) become the ordinary integro-differential equations

$$[C_{i3k3}N'_{kmn} + C_{i3mn} + C_{i3kl}\epsilon_{kls}V_{smn} + B_{i3k3}V'_{kmn}]' = 0,$$

$$
\begin{aligned}
&[D_{i3k3}V'_{kmn} + B_{i3mn} + B_{i3kl}\epsilon_{kls}V_{smn} + B_{i3k3}N'_{kmn}]' = \epsilon_{ijr}(\langle\tilde{C}_{rjmn}\rangle - \tilde{C}_{rjmn}),\\
&\tilde{C}_{rjmn} = C_{rjk3}N'_{kmn} + C_{rjmn} + C_{rjkl}\epsilon_{kls}V_{smn} + B_{ijk3}V'_{kmn},
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&[C_{i3k3}U'_{kmn} + B_{i3mn} + C_{i3kl}\epsilon_{kls}M_{smn} + B_{i3k3}M'_{kmn}]' = 0,\\
&[D_{i3k3}M'_{kmn} + D_{i3mn} + B_{i3kl}\epsilon_{kls}M_{smn} + B_{i3k3}U'_{kmn}]' = \epsilon_{ijr}(\langle\tilde{B}_{rjmn}\rangle - \tilde{B}_{rjmn}),\\
&\tilde{B}_{rjmn} = C_{rjk3}U'_{kmn} + B_{rjmn} + C_{rjkl}\epsilon_{kls}M_{smn} + D_{ijk3}M'_{kmn}.
\end{aligned}
\tag{5.2}
$$

The prime denotes the derivative with respect to x_3, and the angle brackets denote the average value of a function over the plate thickness,

$$
\langle f\rangle \equiv \frac{1}{L}\int_0^L f(x_3)\,dx_3.
$$

The conditions on the plate boundaries become

$$
N_{imn}\big|_{x_3=0,L} = 0,\quad V_{imn}\big|_{x_3=0,L} = 0,\quad M_{imn}\big|_{x_3=0,L} = 0,\quad U_{imn}\big|_{x_3=0,L} = 0.
\tag{5.3}
$$

We integrate the first equation in system (5.1), take into account the relation $\langle N'_{imn}\rangle = 0$, and obtain

$$
\begin{aligned}
N'_{kmn} = {}& [C^{-1}_{k3l3}\langle C^{-1}_{l3p3}\rangle^{-1}\langle C^{-1}_{p3q3}C_{q3mn}\rangle - C^{-1}_{k3q3}C_{q3mn}]\\
&+ [C^{-1}_{k3l3}\langle C^{-1}_{l3p3}\rangle^{-1}\langle C^{-1}_{p3q3}C_{q3ab}(\bullet)\rangle - C^{-1}_{k3q3}C_{q3ab}]\varepsilon_{abs}V_{smn}\\
&+ [C^{-1}_{k3l3}\langle C^{-1}_{l3p3}\rangle^{-1}\langle C^{-1}_{p3q3}C_{q3s3}(\bullet)\rangle - C^{-1}_{k3q3}C_{q3s3}]V'_{smn}.
\end{aligned}
\tag{5.4}
$$

In the last two square brackets, the symbol $(\bullet)$ is replaced by $\varepsilon_{abs}V_{smn}$ and V'_{smn}, respectively. Note that the average values of each row on the right-hand side in (5.4) is zero. Further, we determine $\tilde{C}_{ijmn}$ as

$$
\begin{aligned}
\tilde{C}_{ijmn} = {}& C_{ijmn} + C_{ijk3}C^{-1}_{k3l3}\langle C^{-1}_{l3p3}\rangle^{-1}\langle C^{-1}_{p3q3}C_{q3mn}\rangle - C_{ijk3}C^{-1}_{k3l3}C_{l3mn}\\
&+ [C_{ijab} + C_{ijk3}C^{-1}_{k3l3}\langle C^{-1}_{l3p3}\rangle^{-1}\langle C^{-1}_{p3q3}C_{q3ab}(\bullet)\rangle - C_{ijk3}C^{-1}_{k3q3}C_{q3ab}]\varepsilon_{abs}V_{smn},\\
&+ [B_{ijs3} + C_{ijk3}C^{-1}_{k3l3}\langle C^{-1}_{l3p3}\rangle^{-1}\langle C^{-1}_{p3q3}B_{q3s3}(\bullet)\rangle - C_{ijk3}C^{-1}_{k3q3}B_{q3s3}]V'_{smn}.
\end{aligned}
\tag{5.5}
$$

After integrating the second equation in system (5.1) once, we obtain

$$
\begin{aligned}
V'_{kmn} = {}& -D^{-1}_{k3l3}\big[B_{l3mn} + B_{l3ab}\epsilon_{abs}V_{smn} + B_{l3s3}N'_{smn}\\
&- \langle D^{-1}_{l3p3}\rangle^{-1}\langle D^{-1}_{p3q3}(B_{q3mn} + B_{q3ab}\epsilon_{abs}V_{smn} + B_{q3s3}N'_{smn})\rangle\big]\\
&+ D^{-1}_{k3l3}\Bigg[\epsilon_{ljr}\int_0^{x_3}(\langle\tilde{C}_{rjmn}\rangle - \tilde{C}_{rjmn}(y))\,dy\\
&- \langle D^{-1}_{i3p3}\rangle^{-1}\Big\langle D^{-1}_{p3q3}(z)\epsilon_{qjr}\int_0^{z}(\langle\tilde{C}_{rjmn}\rangle - \tilde{C}_{rjmn}(y))\,dy\Big\rangle\Bigg].
\end{aligned}
\tag{5.6}
$$

The average value of the right-hand side of this expression is zero. Now if we substitute the expressions (5.2) and (5.5) into this formula, then the right-hand side of (5.6) can be written in the same form as the right-hand side of (5.4),

$$
V'_{kmn} = e_{kmn} + \hat{g}_{kab}\epsilon_{abs}V_{smn} + \hat{f}_{ks3}V'_{smn},
\tag{5.7}
$$

$$
e_{kmn} = D^{-1}_{k3s3}T_{smn} - D^{-1}_{k3s3}\langle D^{-1}_{s3p3}\rangle\langle D^{-1}_{p3q3}T_{qmn}\rangle,
\tag{5.8}
$$

$$
T_{imn} = \epsilon_{ijr}\int_0^{x_3}[\langle\tilde{C}^*_{rjmn}\rangle - \tilde{C}^*_{rjmn}(y)]\,dy - (B_{i3mn} + B_{i3k3}r_{kmn}),
\tag{5.9}
$$

$$
\tilde{C}^*_{ijmn} \equiv C_{ijmn} + C_{ijk3}C^{-1}_{k3l3}\langle C^{-1}_{l3p3}\rangle^{-1}\langle C^{-1}_{p3q3}C_{q3mn}\rangle - C_{ijk3}C^{-1}_{k3l3}C_{l3mn},
\tag{5.10}
$$

$$r_{kmn} \equiv C^{-1}_{k3l3}\langle C^{-1}_{l3p3}\rangle^{-1}\langle C^{-1}_{p3q3}C_{q3mn}\rangle - C^{-1}_{k3q3}C_{q3mn}. \tag{5.11}$$

The expressions of $\hat{g}_{kab}$ and $\hat{f}_{ks3}$ are cumbersome operators, and we do not write them out here. Note that the average value of each of the terms in (5.7) is zero. These terms are deviations from the mean value, and the second and third terms are quantities of the order of the squared first term. The main contribution to the expression of the function V_{kmn} is given by the first term. A similar situation also takes place for the function N_{kmn}.

To obtain approximate expressions for the effective characteristics of a layer homogenous over the thickness and made of a Cosserat material, we preserve only the first terms in the expressions for the functions N_{kmn} and V_{kmn}; i.e., we set

$$N'_{kmn} \approx r_{kmn} \quad \Rightarrow \quad N_{kmn} \approx \int\limits_0^{x_3} r_{kmn}(y)\,dy, \tag{5.12}$$

$$V'_{kmn} \approx e_{kmn} \quad \Rightarrow \quad V_{kmn} \approx \int\limits_0^{x_3} e_{kmn}(y)\,dy. \tag{5.13}$$

System (5.2) for the functions U_{kmn} and M_{kmn} can be integrated in a similar way. Omitting insignificant details, we obtain the following approximate expressions for the functions U_{kmn} and M_{kmn}:

$$U'_{kmn} \approx d_{kmn} \quad \Rightarrow \quad U_{kmn} \approx \int\limits_0^{x_3} d_{kmn}(y)\,dy. \tag{5.14}$$

$$M'_{kmn} \approx h_{kmn} \quad \Rightarrow \quad M_{kmn} \approx \int\limits_0^{x_3} h_{kmn}(y)\,dy. \tag{5.15}$$

$$d_{kmn} \equiv C^{-1}_{k3l3}\langle C^{-1}_{l3p3}\rangle^{-1}\langle C^{-1}_{p3q3}B_{q3mn}\rangle - C^{-1}_{k3q3}B_{q3mn}, \tag{5.16}$$

$$h_{kmn} \equiv D^{-1}_{k3s3}\Theta_{smn} - D^{-1}_{k3s3}\langle D^{-1}_{s3p3}\rangle\langle D^{-1}_{p3q3}\Theta_{qmn}\rangle, \tag{5.17}$$

$$\Theta_{imn} = \epsilon_{ijr}\int\limits_0^{x_3}[\langle \tilde{B}^*_{rjmn}\rangle - \tilde{B}^*_{rjmn}(y)]\,dy - (D_{i3mn} + B_{i3k3}d_{kmn}), \tag{5.18}$$

$$\tilde{B}^*_{ijmn} = B_{ijmn} + C_{ijk3}C^{-1}_{k3l3}\langle C^{-1}_{l3p3}\rangle^{-1}\langle C^{-1}_{p3q3}B_{q3mn}\rangle - C_{ijk3}C^{-1}_{k3l3}B_{l3mn}. \tag{5.19}$$

Now we can write out explicit expressions for the effective characteristics of a layer inhomogeneous over the thickness:

$$C^{\text{eff}}_{ijmn} \approx \langle \tilde{C}_{ijmn}\rangle = \langle C_{ijmn} + C_{ijkl}(\delta_{i3}N'_{kmn} + \epsilon_{kls}V_{smn}) + B_{ijk3}V'_{kmn}\rangle, \tag{5.20}$$

$$D^{\text{eff}}_{ijmn} \approx \langle \tilde{D}_{ijmn}\rangle = \langle D_{ijmn} + B_{ijkl}(\delta_{i3}U'_{kmn} + \epsilon_{kls}M_{smn}) + D_{ijk3}M'_{kmn}\rangle, \tag{5.21}$$

$$\begin{aligned} B^{\text{eff}}_{ijmn} &\approx \langle \tilde{B}_{ijmn}\rangle = \langle \tilde{\tilde{B}}_{ijmn}\rangle = \langle B_{ijmn} + B_{ijkl}(\delta_{i3}N'_{kmn} + \epsilon_{kls}V_{smn}) + D_{ijk3}V'_{kmn}\rangle \\ &= \langle B_{ijmn} + C_{ijkl}(\delta_{i3}U'_{kmn} + \epsilon_{kls}M_{smn}) + B_{ijk3}U'_{kmn}\rangle. \end{aligned} \tag{5.22}$$

ACKNOWLEDGMENTS

The research was supported by the Russian Foundation for Basic Research, project No. 12-01-00020a.

REFERENCES

1. Yu. N. Rabotnov, *Mechanics of Deformable Solids* (Nauka, Moscow, 1979) [in Russian].
2. Yu. N. Rabotnov, *Elements of Hereditary Mechanics of Solids* (Nauka, Moscow, 1977) [in Russian].
3. Yu. N. Rabotnov, *Creep of Structural Members* (Nauka, Moscow, 1966) [in Russian].
4. N. S. Bakhvalov, "Averaged Characteristics of Bodies with Periodic Structure," Dokl. Akad. Nauk SSSR **218** (5), 1046–1048 (1974) [Soviet Phys. Dokl. (Engl. Transl.) **9** (10), 650–651 (1975)].

5. N. S. Bakhvalov, "Homogenization of Partial Differential Equations with Rapidly Oscillating Coefficients," Dokl. Akad. Nauk SSSR **221** (3), 516–519 (1975).
6. N. S. Bakhvalov and G. P. Panasenko, *Homogenisation: Averaging Processes in Periodic Media* (Nauka, Moscow, 1984) [in Russian].
7. B. E. Pobedrya, *Mechanics of Composite Materials* (Izd-vo MGU, Moscow, 1984) [in Russian].
8. V. L. Berdichevskii, *Variational Principles of Continuum Mechanics* (Nauka, Moscow, 1983) [in Russian].
9. E. Sanchez-Palencia, *Nonhomogeneous Media and Vibration Theory* (Springer, Berlin, 1980; Mir, Moscow, 1984).
10. I. V. Andrianov, V. A. Lesnichaya, and L. I. Manevich, *Averaging Method in Statics and Dynamics of Ribbed Shells* (Nauka, Moscow, 1985) [in Russian].
11. A. L. Kalamkarov, *Composite and Reinforced Elements of Construction* (Wiley John, Baffins Lane, Chechester, West Sussex PO19, England, 1992).
12. A. B. Movchan, N. V. Movchan, and C. G. Poulton, *Asymptotic Models of Fields in Dilute and Densely Packed Composites* (Imperial College Press, London, 2002).
13. D. I. Bardzokas and A. I. Zobnin, *Mathematical Modeling of Physical Processes in Composite Materials of Periodic Structure* (Editorial URSS, Moscow, 2003) [in Russian].
14. V. I. Bolshakov, I. V. Andrianov, and V. V. Danishevskii, *Asymptotic Methods for Calculating Composite Materials with Periodic Structure Taken into Account* (Porogi, Dnepropetrovsk, 2008) [in Russian].
15. G. B. Kolchin and E. A. Faverman, *Theory of Elasticity of Inhomogeneous Bodies. Bibliographic Index of Domestic and Foreign Literature* (Shtiintsa, Kishinev, 1972) [in Russian].
16. G. B. Kolchin and E. A. Faverman, *Theory of Elasticity of Inhomogeneous Bodies. Bibliographic Index of Domestic and Foreign Literature for 1970–1973* (Shtiintsa, Kishinev, 1977) [in Russian].
17. V. A. Lomakin, *Theory of Elasticity of Inhomogeneous Bodies* (Izd-vo MGU, Moscow, 1976) [in Russian].
18. S. G. Lekhnitskii, *Torsion of Anisotropic and Inhomogeneous Rods* (Nauka, Moscow, 1971) [in Russian].
19. V. I. Gorbachev, "Green Tensor Method for Solving Boundary Value Problems of the Theory of Elasticity for Inhomogeneous Media," Vych. Mekh. Sploshn. Sred No. 2, 61–76 (1991).
20. V. I. Gorbachev, *A Version of the Averaging Method for Solving Boundary-Value Problems of Inhomogeneous Elasticity*, Doctoral Dissertation in Physics and Mathematics (Lomonosov Moscow State Univ., Moscow, 1991).
21. V. I. Gorbachev and A. S. Kokarev, "Integral Formula in Dynamic Problem of Inhomogeneous Elasticity," Vestnik Moskov. Univ. Ser. I. Mat. Mekh., No. 2, 62–66 (2005).
22. V. I. Gorbachev, "Dynamic Problems of Composite Mechanics," Izv. Ross. Akad. Nauk. Ser. Fiz. **75** (1), 117–122 (2011) [Bull. Russ. Acad. Sci. Phys. (Engl. Transl.) **75** (1), 110–115 (2011)].
23. V. I. Gorbachev, "Integral formulas in Symmetric and Asymmetric Elasticity," Vestnik Moskov. Univ. Ser. I. Mat. Mekh. **64** (6), 57–60 (2009) [Moscow Univ. Mech. Bull. (Engl. Transl.) **64** (6), 148–151 (2009)].
24. W. Nowacki, *Dynamic Problems of Thermal Elasticity* (Mir, Moscow, 1970) [in Russian].
25. V. I. Gorbachev and A. N. Emel'yanov, "Homogenization of Problems of Cosserat Theory of Elasticity of Composites," in *Additional Materials. Intern. Scientific Symposium in Problems of Mechanics of Deformable Solids Dedicated to A. A. Il'yushin on the Occasion of His 100th Birthday, January 20–21, 2011* (Izd-vo MGU, Moscow, 2012), pp. 81–88 [in Russian].
26. Z. Hashin and B. W. Rosen, "The Elastic Moduli of Fiver-Reinforced Materials," J. Appl. Mech. **31** (2), 223–232 (1964).
27. V. I. Gorbachev and L. V. Olekhova, "Effective Properties of a Nonuniform Beam under Torsion," Vestnik Moskov. Univ. Ser. I. Mat. Mekh. **62** (5), 41–48 (2007) [Moscow Univ. Mech. Bull. (Engl. Transl.) **62** (5), 123–130 (2007)].
28. E. I. Grigolyuk and L. A. Filshtinskii, *Perforated Plates and Shells* (Nauka, Moscow, 1970) [in Russian].
29. E. I. Grigolyuk and L. A. Filshtinskii, *Regular Piecewise Homogeneous Structure with Defects* (Fizmatlit, Moscow, 1994) [in Russian].
30. G. A. Vanin, *Micromechanics of Composite Materials* (Naukova Dumka, Kiev, 1985) [in Russian].
31. L. V. Olekhova, *Torsion of Inhomogeneous Anisotropic Rod*, Master's Thesis (Lomonosov Moscow State Univ., Moscow, 2009).