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About a Problem of Sturm—Liuvill

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Abstract—The linear ordinary differential equation of the second order with variable coefficients (initial equation) is considered. Its common solution, containing two arbitrary constant, is by means of the integral formula offered earlier in works of the author. This solution is used in a problem of Sturm—Liuvill by definition of eigenvalues of an initial equation under various homogeneous conditions.

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1. INTRODUCTION

Many processes happening in inhomogeneous skew fields, in particular in aggregates, are described by linear differential equations with the variable coefficients depending on coordinates. For allocation unique solution of such equations are put boundary and entry conditions. If the equation is homogeneous, and boundary and entry conditions zero such problem has trivial, that is a trivial solution. However such situation when along with a trivial solution there is also a solution distinct from the zero is possible. Usually this situation arises in that case when one, or some coefficients of the equation depend on numerical parameter λ. The same parameter λ can enter and in homogeneous boundary and entry conditions. In these cases there can be such values of parameter λ at which the problem has also a nontrivial solution. Such values λ are called as eigenvalues, and solutions of the equation for each of eigenvalues are called as eigenfunctions. The described general situation is called as an eigenvalue problem [1], [2, pp. 24—32]. Problems on definition of eigenvalues arise in the mechanic, the physicist, a quantum mechanics and in other most different cases. For example at attempt to discover a solution parabolic, or the hyperbolic equations a method of a separation of variables. At study of movement of the elementary particle which are in a potential field. In the given work we will consider the self-conjugate eigenvalue problem of an ordinary differential equation of the second order with variable coefficients. It is accepted to name this problem a problem of Sturm—Liuvill. The theory of a problem of Sturm—Liuvill in the accessible form is stated in books [3, p. 546], [4, pp. 568—609], [5, p. 261]. Inverse problems of Sturm—Liuvill were considered in work [6].

2. THE BASIC CONCEPTS

2.1. Initial Equation. Eigenvalues and Eigenfunctions

A homogeneous second-order self-adjoint differential equation with variable coefficients is considered:

$$\frac{d}{dx}\left[C(x)\frac{du}{dx}\right] + q(x,\lambda)u = 0, \quad q(x,\lambda) = h(x) + \lambda\varrho(x), \quad x \in (a,b). \tag{1}$$

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Such equations are called initial equations. Parameter λ is a constant number. It is supposed that equation coefficients (1) are the limited integrable functions of coordinate x. Coefficient $C(x)$ is positive, integrated on $[a, b]$ function which also can depend on parameter λ. For the equation (1) the boundary value problem with homogeneous conditions is considered

$$\alpha_1 u(a) + \beta_1 u'(a) = 0, \quad \alpha_2 u(b) + \beta_2 u'(b) = 0, \tag{2}$$

where α_1, α_2, β_1, β_2 are constants. Depending on a choice of constants $\alpha_1, \beta_1, \alpha_2, \beta_2$ from conditions (2) turns out the first, second and third boundary value problems [3, p. 546]. It is possible to consider also a Cauchy problem when on one edge are set value of required function $u(x)$ and its first derivative $u'(x)$ is the entry conditions. The even more general conditions are, so called, not breaking up boundary conditions [6]. The general view which can be noted by formulas

$$\begin{aligned} &\alpha_{11} u(a) + \alpha_{12} u(b) + \beta_{11} u'(a) + \beta_{12} u'(b) = 0, \\ &\alpha_{21} u(a) + \alpha_{22} u(b) + \beta_{21} u'(a) + \beta_{22} u'(b) = 0. \end{aligned} \tag{3}$$

2.2. The Accompanying Equation and its Common Solution

Let's consider function $v(x)$ which represents a solution of a homogeneous equation of the same type, as initial, but with constant coefficients $C_o = const > 0$ and $q_o = const$,

$$C_o \frac{d^2 v}{dx^2} + q_o v = 0, \quad x \in (a, b). \tag{4}$$

Such equation we will name the accompanying equation. The common solution of the accompanying equation (4) looks like:

$$v(x) = K_1 e^{i\mu_o x} + K_2 e^{-i\mu_o x}, \quad \begin{cases} \mu_o = \sqrt{\frac{q_o}{C_o}}, & q_o > 0, \\ \mu_o = i\sqrt{\frac{-q_o}{C_o}}, & q_o < 0, \end{cases} \tag{5}$$

where i is complex unit, K_1 and K_2 are arbitrary complex constants. The valid solution of the accompanying equation (4) is represented different formulas depending on a coefficient sign q_o

$$v(x) = \begin{cases} K_1 \cos \mu_o x + K_2 \sin \mu_o x, \quad \mu_o = \sqrt{\frac{q_o}{C_o}} \quad \text{if} \quad q_o > 0, \\ K_1 x + K_2, \quad \mu_o = 0 \quad \text{if} \quad q_o = 0, \\ K_1 e^{-\mu_o x} + K_2 e^{\mu_o x}, \quad \mu_o = \sqrt{\frac{-q_o}{C_o}} \quad if \quad q_o < 0. \end{cases} \tag{6}$$

Here K_1 and K_2 are arbitrary valid constants.

2.3. Note about Coefficients of the Accompanying Equation

Let's notice that in the accompanying equation coefficients C_o and q_o within the limits of the specified restrictions — absolutely arbitrary constants.

1. In particular the coefficient q_o can be accepted equal to zero. In this case the common solution of the accompanying equation is a linear function of coordinate and looks like (6).

2. It is possible to consider that q_o there is a nonzero constant not dependent from λ.

3. It is possible to assume that q_o there is a function from λ.

4. Moreover, it is possible to assume that both coefficients C_o and q_o are the functions of coordinate and parameter λ. In the latter case the common solution of the accompanying equation any more it is not represented formulas (5), (6).

5. In the present work we will suppose accompanying coefficients C_o and q_o equal to effective performances [7, 8] so that

$$C_o = \frac{1}{\langle 1/C(x) \rangle}, \quad q_o(\lambda) = \langle q(x, \lambda) \rangle, \quad f_o \equiv \langle f(x) \rangle \equiv \frac{1}{b-a} \int_a^b f(x) dx.$$

3. THE INTEGRAL FORMULA OF REPRESENTATION OF A SOLUTION OF AN INITIAL EQUATION THROUGH A SOLUTION OF THE ACCOMPANYING EQUATION

3.1. The Integral Formula

In works [9, 10] the integral formula of representation of a solution of an initial equation (1) through a solution of the accompanying equation by means of fundamental function $G(x,\xi)$ an initial equation is received

$$u(x) = v(x) + \int\limits_a^b \frac{dG(x,\xi)}{d\xi}\tilde{C}(\xi)v'(\xi)d\xi - \int\limits_a^b G(x,\xi)\tilde{q}(\xi,\lambda)v(\xi)d\xi. \tag{7}$$

Here $\tilde{C}(\xi) = C_o - C(\xi)$, $\tilde{q}(\xi,\lambda) = q_o - q(\xi,\lambda)$. Fundamental function of an initial equation (1) is understood as any function $G(x,\xi)$, satisfying to a following fundamental equation:

$$\frac{d}{dx}\left[C(x)\frac{dG(x,\xi)}{dx}\right] + q(x,\lambda)G(x,\xi) + \delta(x-\xi) = 0, \quad x,\xi \in (a,b), \tag{8}$$

where $\delta(x-\xi)$ is the generalized Dirac delta-function [11, p. 194], [12, p. 42]. Through coefficient $q(x,\lambda)$ the parameter λ is included into fundamental function $G(x,\xi)$. Hardly it will be more low offered two modes of an analytical solution of the equation (8) in the form of rows.

3.2. The Initial Equation Common Solution

Let's substitute the common solution (5) the accompanying equation in the integral formula (7) also we will receive the common solution of an initial homogeneous equation of Sturm–Liuvill (1)

$$u(x) = K_1 A(x,\lambda) + K_2 B(x,\lambda), \tag{9}$$

where K_1 and K_2 are the arbitrary complex constants. Coefficients $A(x,\lambda)$ and $B(x,\lambda)$ are also complex magnitudes

$$A(x,\lambda) = e^{i\mu_o x} + i\mu_o \int\limits_a^b G'_\xi(x,\xi)\tilde{C}(\xi)e^{i\mu_o\xi}d\xi - \int\limits_a^b G(x,\xi)\tilde{q}(\xi)e^{i\mu_o\xi}d\xi,$$

$$B(x,\lambda) = e^{-i\mu_o x} - i\mu_o \int\limits_a^b G'_\xi(x,\xi)\tilde{C}(\xi)e^{-i\mu_o\xi}d\xi - \int\limits_a^b G(x,\xi)\tilde{q}(\xi)e^{-i\mu_o\xi}d\xi. \tag{10}$$

The valid solution has the same appearance (9), thus K_1 and K_2 are the valid constants. Real valued functions $A(x,\lambda)$ and $B(x,\lambda)$ have the various aspect depending on a coefficient sign q_o:

$$A(x,\lambda) = \begin{cases} \cos(\mu_o x) - \mu_o \int_a^b G'_\xi(x,\xi)\tilde{C}(\xi)\sin(\mu_o\xi)d\xi - \int_a^b G(x,\xi)\tilde{q}(\xi)\cos(\mu_o\xi)d\xi, \quad q_o > 0, \\ x + \int_a^b G'_\xi(x,\xi)\tilde{C}(\xi)d\xi + \int_a^b \xi G(x,\xi)q(\xi)d\xi, \quad q_o = 0, \\ e^{-\mu_o x} - \mu_o \int_a^b G'_\xi(x,\xi)\tilde{C}(\xi)e^{-\mu_o\xi}d\xi - \int_a^b G(x,\xi)\tilde{q}(\xi)e^{-\mu_o\xi}d\xi, \quad q_o < 0; \end{cases} \tag{11}$$

$$B(x,\lambda) = \begin{cases} \sin(\mu_o x) + \mu_o \int_a^b G'_\xi(x,\xi)\tilde{C}(\xi)\cos(\mu_o\xi)d\xi - \int_a^b G(x,\xi)\tilde{q}(\xi)\sin(\mu_o\xi)d\xi, \quad q_o > 0, \\ 1 + \int_a^b G(x,\xi)q(\xi)d\xi, \quad q_o = 0, \\ e^{\mu_o x} + \mu_o \int_a^b G'_\xi(x,\xi)\tilde{C}(\xi)e^{\mu_o\xi}d\xi - \int_a^b G(x,\xi)\tilde{q}(\xi)e^{\mu_o\xi}d\xi, \quad q_o < 0. \end{cases} \tag{12}$$

K_1 and K_2 it is necessary to search for nonzero valid constants from two conditions (2), or (3). As a result it turns out complicated enough the nonlinear equation for an eigenvalue λ, at which homogeneous equation with homogeneous conditions the nontrivial solution has Sturm–Liuvill.

4. METHOD OF PERTURBATIONS FOR DEFINITION OF A FUNDAMENTAL SOLUTION OF AN INITIAL EQUATION

The initial equation common solution is under formulas (9) if it is known fundamental solution of the equation (8) with variable coefficients. However problem of search of an exact analytical solution of the fundamental equation, generally associations of coefficients on coordinate, it is hardly solvable. We will search for fundamental function method of perturbations [13, 14].

4.1. The Scheme of a Method of Perturbations

In a method of perturbations the equation (8) corresponds as follows (association of coefficient q from λ is lowered):

$$\frac{d}{dx}\left[C(x)\frac{dG(x,\xi)}{dx}\right]+\varkappa q(x)G(x,\xi)+\delta(x-\xi)=0, \tag{13}$$

where $\varkappa$ is the revolting parameter which in a final output we will suppose the equal to unit. We will search for an equation solution (13) in the form of a series on parameter degrees

$$G(x,\xi,\varkappa)=\sum_{n=0}^{\infty}\varkappa^n G_n(x,\xi). \tag{14}$$

Let's substitute a series (14) in the equation (13), we will collect coefficients at identical degrees $\varkappa$ also we will equate them to zero. It is as a result received the recurrent sequence of the equations

$$\left[C(x)G_0'(x,\xi)\right]'+\delta(x-\xi)=0, \quad \left[C(x)G_n'(x,\xi)\right]'+q(x)G_{n-1}(x,\xi)=0, \quad n>0,$$

or

$$G_0(x,\xi)=-\int_a^x \frac{h(z-\xi)}{C(z)}dz, \quad G_n(x,\xi)=-\int_a^x \frac{dx_1}{C(x_1)}\int_a^{x_1} q(x_2)G_{n-1}(x_2,\xi)dx_2. \tag{15}$$

Let's substitute a series (14) in the equation (13), we will collect coefficients at identical degrees $\varkappa$ also we will equate them to zero. It is as a result received the recurrent sequence of the equations $G_0(x,\xi)$

$$G_0(x,\xi)=-\int_a^x \frac{h(z-\xi)}{C(z)}dz=-\begin{cases}0, & x<\xi\\ \int\limits_\xi^x \frac{dz}{C(z)}, & x\geqslant\xi\end{cases}=-h(x-\xi)\int_\xi^x \frac{dz}{C(z)}.$$

After that from a recurrence formula (15) sequentially there are all remaining functions $G_n(x,\xi)$

$$\begin{aligned}G_n(x,\xi)&=(-1)^{n+1}\int_a^x \frac{dx_1}{C(x_1)}\int_a^{x_1} q(x_2)dx_2\cdots\int_a^{x_{2n-2}}\frac{dx_{2n-1}}{C(x_{2n-1})}\int_a^{x_{2n-1}} q(x_{2n})dx_{2n}\int_a^{x_{2n}}\frac{h(z-\xi)dz}{C(z)}\\ &=(-1)^{n+1}h(x-\xi)\int_\xi^x \frac{dx_1}{C(x_1)}\int_\xi^{x_1} q(x_2)dx_2\cdots\int_\xi^{x_{2n-2}}\frac{dx_{2n-1}}{C(x_{2n-1})}\int_\xi^{x_{2n-1}} q(x_{2n})dx_{2n}\int_\xi^{x_{2n}}\frac{dz}{C(z)}\\ &=(-1)^{n+1}h(x-\xi)\int_\xi^x \psi(x_1,\xi)dx_1\cdots\int_\xi^{x_{n-1}}\psi(x_n,\xi)dx_n\int_\xi^{x_n}\frac{dz}{C(z)}.\end{aligned} \tag{16}$$

Here for entry reduction the auxiliary label is introduced

$$\psi(x,\xi)=\frac{1}{C(x)}\int_\xi^x q(y)dy.$$

A series for fundamental function will become (14), in which $\varkappa = 1$, that is $G(x,\xi) = \sum_{n=0}^{\infty} G_n(x,\xi)$. Derivatives on change ξ from $G_n(x,\xi)$ will be necessary for us. At $n = 0$ it is received

$$\frac{\partial G_0(x,\xi)}{\partial \xi} = -\frac{\partial}{\partial \xi}\left[h(x-\xi)\int\limits_{\xi}^{x}\frac{dz}{C(z)}\right] = \underline{\delta(x-\xi)\int\limits_{\xi}^{x}\frac{dz}{C(z)}} + \frac{h(x-\xi)}{C(\xi)} = \frac{h(x-\xi)}{C(\xi)}. \tag{17}$$

Function $G_0(x,\xi)$ and its derivatives enter under integrals in formulas (7) and (10)–(12). Therefore composed with Dirac delta-function in the formula (17) it is lowered from final outputs. At $n > 0$ of expression for derivative from $G_n(x,\xi)$ follow from a recurrence formula (16)

$$\frac{\partial G_n(x,\xi)}{\partial x} = (-1)^{n+1}\frac{h(x-\xi)}{C(x)}\int\limits_{\xi}^{x} q(x_1)dx_1\int\limits_{\xi}^{x_1}\psi(x_2,\xi)dx_2\cdots\int\limits_{\xi}^{x_{n-1}}\psi(x_n,\xi)dx_n\int\limits_{\xi}^{x_n}\frac{dz}{C(z)}, \tag{18}$$

$$\frac{\partial G_n(x,\xi)}{\partial \xi} = (-1)^{n}\frac{h(x-\xi)}{C(\xi)}\int\limits_{\xi}^{x}\psi(x_1,\xi)dx_1\cdots\int\limits_{\xi}^{x_{n-1}}\psi(x_n,\xi)dx_n. \tag{19}$$

4.2. Special Case

Let $q(x) = \lambda\varrho(x)$, where $\varrho(x) > 0$. In this case $\psi(x,\xi) = \lambda\bar{\psi}$, $\bar{\psi} \equiv \frac{1}{C(x)}\int\limits_{\xi}^{x}\varrho(y)dy$, and formulas (16), (18), (19) become

$$G_n(x,\xi) = (-1)^{n+1}\lambda^n h(x-\xi)\int\limits_{\xi}^{x}\bar{\psi}(x_1,\xi)dx_1\cdots\int\limits_{\xi}^{x_{n-1}}\bar{\psi}(x_n,\xi)dx_n\int\limits_{\xi}^{x_n}\frac{dz}{C(z)}, \tag{20}$$

$$\frac{\partial G_n(x,\xi)}{\partial x} = (-1)^{n+1}\lambda^n\frac{h(x-\xi)}{C(x)}\int\limits_{\xi}^{x}\varrho(x_1)dx_1\int\limits_{\xi}^{x_1}\bar{\psi}(x_2,\xi)dx_2\cdots\int\limits_{\xi}^{x_{n-1}}\bar{\psi}(x_n,\xi)dx_n\int\limits_{\xi}^{x_n}\frac{dz}{C(z)},$$

$$\frac{\partial G_n(x,\xi)}{\partial \xi} = (-1)^{n}\lambda^n\frac{h(x-\xi)}{C(\xi)}\int\limits_{\xi}^{x}\bar{\psi}(x_1,\xi)dx_1\cdots\int\limits_{\xi}^{x_{n-1}}\bar{\psi}(x_n,\xi)dx_n. \tag{21}$$

The coefficient q_o becomes the accompanying equation $q_o(\lambda) = \lambda\varrho_o$, $\varrho_o = \langle\varrho(x)\rangle$.

5. THE SPECTRAL EQUATIONS FOR EIGENVALUES

The common solution of an initial equation with two arbitrary constants is presented by the formula (9). To deduce the equation for an eigenvalue λ are necessary homogeneous boundary conditions. The general and which general view is given by formulas (2) and (3). We will consider in the beginning the most simple cases.

5.1. The Spectral Equation of the First Boundary Value Problem

In case of the first boundary value problem it is necessary in conditions (2) to suppose $\alpha_1 = \alpha_2 = 1$, and $\beta_1 = \beta_2 = 0$. Then on the segment extremities $[a,b]$ a solution of the initial equations (2) should be zero, that is $u(a) = u(b) = 0$. From here and from the initial equation common solution (9) we receive a set of equations

$$\begin{cases} K_1A(a,\lambda) + K_2B(a,\lambda) = 0, \\ K_1A(b,\lambda) + K_2B(b,\lambda) = 0. \end{cases} \tag{22}$$

Constants K_1 and K_2 will be distinct from zero if the system determinant (22) is equal to zero. From here we receive the spectral equation of the first boundary value problem

$$A(a,\lambda)B(b,\lambda) - A(b,\lambda)B(a,\lambda) = 0. \tag{23}$$

5.2. The Spectral Equation of the Second Boundary Value Problem

$$C(x)u'(x) = K_1C(x)A'(x) + K_2C(x)B'(x).$$

Homogeneous boundary conditions of the second boundary value problem look like

$$C(a)u'(a) = C(b)u'(b) = 0.$$

The set of equations implies from these conditions for constants K_1 and K_2

$$\begin{cases} K_1C(a)A'(a,\lambda) + K_2C(a)B'(a,\lambda) = 0, \\ K_1C(b)A'(b,\lambda) + K_2C(b)B'(b,\lambda) = 0. \end{cases}$$

From here the spectral equation of the second boundary value problem follows

$$\big[C(x)A'(x,\lambda)\big]_{x=a}\big[C(x)B'(x,\lambda)\big]_{x=b} - \big[C(x)A'(x,\lambda)\big]_{x=b}\big[C(x)B'(x,\lambda)\big]_{x=a} = 0. \quad (24)$$

5.3. The Spectral Equation of the Mixed Boundary Value Problem

In the mixed problem, at the end of $x = a$ a zero value of the function $u(x)$ is given, and at the other end the product $C(x)u'(x)$ is set to zero: $u(a) = 0$, $C(b)u'(b) = 0$. As a result the spectral equation of the third boundary value problem looks like:

$$A(a,\lambda)\big[C(x)B'(x,\lambda)\big]_{x=b} - \big[C(x)A'(x,\lambda)\big]_{x=b}B(a,\lambda)\big] = 0. \quad (25)$$

5.4. The General Spectral Equation in a Boundary Value Problem

Homogeneous boundary conditions of a general view (2) are noted by the following formulas:

$$\begin{cases} K_1\Big[\alpha_1A(a,\lambda) + \beta_1A'(a,\lambda)\Big] + K_2\Big[\alpha_1B(a,\lambda) + \beta_1B'(a,\lambda)\Big] = 0, \\ K_1\Big[\alpha_2A(b,\lambda) + \beta_2A'(b,\lambda)\Big] + K_2\Big[\alpha_2B(b,\lambda) + \beta_2B'(b,\lambda)\Big] = 0. \end{cases}$$

From here the spectral equation of a general view follows

$$\Big[\alpha_1A(a,\lambda) + \beta_1A'(a,\lambda)\Big]\Big[\alpha_2B(b,\lambda) + \beta_2B'(b,\lambda)\Big]$$
$$- \Big[\alpha_2A(b,\lambda) + \beta_2A'(b,\lambda)\Big]\Big[\alpha_1B(a,\lambda) + \beta_1B'(a,\lambda)\Big] = 0. \quad (26)$$

5.5. The Spectral Equation in a Cauchy Problem

In this case on the left extremity $x = a$ zero values of a solution of an initial equation and it are set derivative on coordinate. Therefore the spectral equation in a Cauchy problem will be the following:

$$A(a,\lambda)\big[C(x)B'(x,\lambda)\big]_{x=a} - \big[C(x)A'(x,\lambda)\big]_{x=a}B(a,\lambda) = 0. \quad (27)$$

5.6. About the Spectral Equations

In the spectral equations (23), (24), (25), (26) and (27) functions $A(x,\lambda)$ and $B(x,\lambda)$ which are defined on participate to formulas (11) and (12). Into these formulas enters fundamental solution of an initial equation which, in most cases, is in the form of a series. If the expression for of the fundamental solution is bounded by a finite sum $G_{(I)}(x,\xi) = \sum_{n=0}^{I} G_n(x,\xi)$, then the spectral equations are approximate equations and instead of $A(x,\lambda)$ and $B(x,\lambda)$ they include approximate values of $A_{(I)}(x,\lambda)$ and $B_{(I)}(x,\lambda)$. These functions are defined on to formulas (11) and (12), in which fundamental function $G(x,\xi)$ and its derivative $G'_\xi(x,\xi)$ are substituted with the final sums $G_{(I)}(x,\xi)$ and it derivative, accordingly

$$A_{(I)}(x,\lambda)=\begin{cases}\cos(\mu_0 x)-\mu_0\int\limits_a^b[G_{(I)}]'_\xi\tilde{C}(\xi)\sin(\mu_0\xi)d\xi-\int\limits_a^b G_{(I)}\tilde{q}(\xi)\cos(\mu_0\xi)d\xi, & q_0>0,\\ x+\int\limits_a^b[G_{(I)}]'_\xi\tilde{C}(\xi)d\xi+\int\limits_a^b\xi G_{(I)}q(\xi)d\xi, & q_0=0,\\ e^{-\mu_0 x}-\mu_0\int\limits_a^b[G_{(I)}]'_\xi\tilde{C}(\xi)e^{-\mu_0\xi}d\xi-\int\limits_a^b G_{(I)}\tilde{q}(\xi)e^{-\mu_0\xi}d\xi, & q_0<0;\end{cases}$$

$$B_{(I)}(x,\lambda)=\begin{cases}\sin(\mu_0 x)+\mu_0\int\limits_a^b[G_{(I)}]'_\xi\tilde{C}(\xi)\cos(\mu_0\xi)d\xi-\int\limits_a^b G_{(I)}\tilde{q}(\xi)\sin(\mu_0\xi)d\xi, & q_0>0,\\ 1+\int\limits_a^b G_{(I)}q(\xi)d\xi, & q_0=0,\\ e^{\mu_0 x}+\mu_0\int\limits_a^b[G_{(I)}]'_\xi\tilde{C}(\xi)e^{\mu_0\xi}d\xi-\int\limits_a^b G_{(I)}\tilde{q}(\xi)e^{\mu_0\xi}d\xi, & q_0<0;\end{cases}$$

$$G_{(I)}(x,\xi)=\sum_{n=0}^{I}G_n(x,\xi),\qquad\frac{\partial G_{(I)}(x,\xi)}{\partial\xi}=\sum_{n=0}^{I}\frac{\partial G_n(x,\xi)}{\partial\xi},$$

where $G_n(x,\xi)$ and $\frac{\partial G_n(x,\xi)}{\partial\xi}$ are under formulas (20) and (21).

6. CONCLUSION

The classical problem of Sturm–Liuvill for the homogeneous self-interfaced ordinary is considered differential equation of the second order with the variable coefficients depending on the constant parameter. It is shown that by means of the integral formula received earlier by the author, it is possible to write out the equation common solution, containing two arbitrary constant. The various possible are written out homogeneous boundary and entry conditions from which the nonzero, basically, should be defined values of these constants. For various homogeneous conditions the general view of the spectral equations is received for eigenvalues of the parameter entering into coefficients.

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