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Three-Dimensional Bayesian Inversion of Audiomagnetotelluric Data in the Salinity Zone of a Coastal Groundwater Reservoir

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Abstract—The applicability of the Bayesian inversion of audiomagnetotelluric data to the electromagnetic mapping of groundwater reservoirs is examined. A model example is used to demonstrate that both the reservoirs and their salinity zones can be mapped using measurements of magnetotelluric fields in the audio frequency range.

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INTRODUCTION

The problem of salinization/contamination of fresh water of lakes and other reservoirs is becoming more relevant, which emphasizes the validity of development of effective methods for their spatial mapping and monitoring of their salinity level. Since the electrical conductivity of fresh water depends on its salinity level, a possible way of solving this problem is magnetotelluric (MT) monitoring of the water electrical conductivity. This approach can vary in its details depending on the chosen object of study, but its basic applicability can be demonstrated with the model problem described below.

FORMULATION OF THE PROBLEM

The goal is to use data of audiomagnetotelluric (AMT) sounding for the detection of groundwater salinization by seawater and the possible mapping of the saline aquifer. A geoelectric model illustrating the pertinent problem is shown in Fig. 1. The values σ_{FW} = 0.05 S/m and σ_{SW} = 0.4 W/m are the electrical conductivities of ground saturated with fresh and saline water, respectively. The conductivity of seawater was set equal to σ_S = 4 S/m.

Data

The model fields were calculated with the use of the FDM3D software package [Spichak, 1983] at periods of 0.005, 0.01, and 0.02 s for two polarizations of the initial field. Since actual data can only be measured at the Earth's surface, only land values of simulated electric fields (in the area bounded by surface projections of horizontal boundaries of the search zone) were used for the inversion.

A Priori Information

Solving the problem formulated above, we assumed that

(1) the horizontally layered structure of the electrical conductivity in the study region is known;

(2) the possible salinization zone of groundwater has boundaries shown by the dashed line in Fig. 1;

(3) within the search zone, the electrical conductivity of water can assume, with equal probability, the values σ_{FW} = 0.05 S/m and σ_{SW} = 0.4 W/m in accordance with the uncertainty of expert estimation of the salinity at a given point of the zone.

BAYESIAN METHOD OF STATISTICAL INVERSION

The Bayesian statistical approach to geophysical data inversion was proposed for the first time by Tarantola [1987]. Within the framework of this approach, an effective method was developed in [Spichak, 1999a] for reducing the solution of the theoretically nonunique inverse problem to the calculation of uncertainty estimates for an a posteriori distribution of the sought function. Below, we present the main features of the algorithm using the 3-D electromagnetic data inversion as an example [Spichak, 1999b].

Let y_{ij} be values of the transforms $h(\mathbf{E}(M_i, \omega_j, \sigma)$ and $\mathbf{H}(M_i, \omega_i, \sigma)$ be values of the fields measured at given surface points M_i (i = 1, 2, …, I) at frequencies ω_j (j = 1, 2, …, J). For each point M_i and each frequency ω_j, we can write

$$y_{ij} = h((\mathbf{E}(M_i, \omega_j, \sigma), \mathbf{H}(M_i, \omega_j, \sigma)) + w_{ij}, \quad (1)$$

where w_{ij} (i = 1, …, I; j = 1, …, J) is the noise matrix regarded as the realization of an independent random

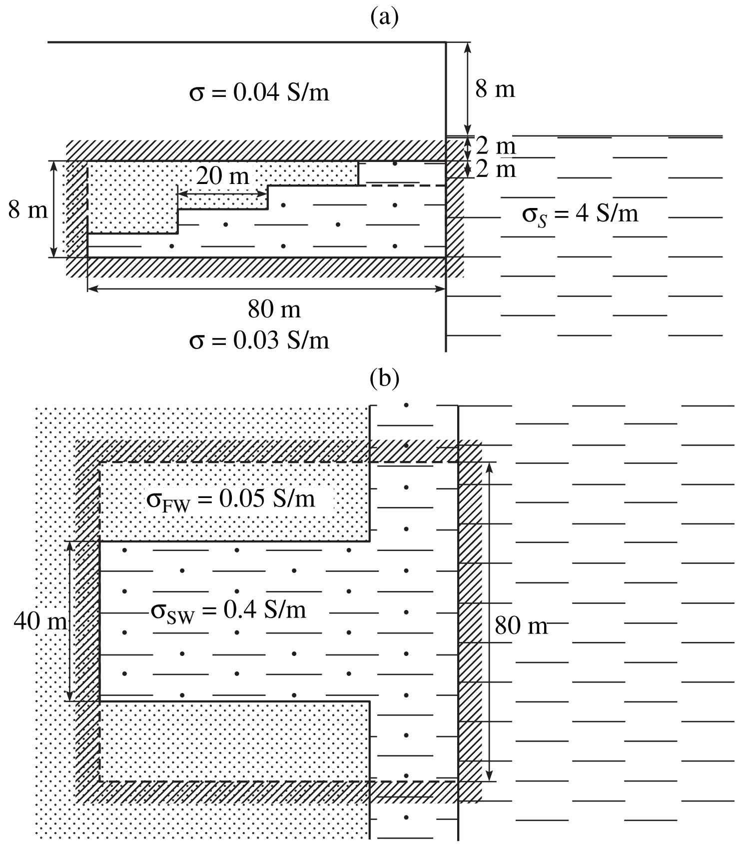

Fig. 1. The geoelectric model corresponding to the formulation of the problem of the groundwater salinization by seawater: (a) vertical cross section in the symmetry plane; (b) horizontal cross section of the lower layer in the search region whose boundary is shown by the dashed line.

variable with the probability density function (PDF) p_{ij}, the variance δ_{ij}, and a zero mean.

Using measured data and a priori information, we have to estimate the components of the vector of a posteriori values defining the sought function of electrical conductivity $\sigma = \sigma_k$ ($k = 1, 2, \ldots, K$), where K is the number of homogeneous subregions of the search zone.

In terms of the Bayesian approach, a priori estimates and hypotheses are easily incorporated in the inversion process through a probabilistic law that is called the a priori probability density function defined on the set of possible values of the sought function. For definiteness, we assume that the set of all possible values that can be taken on by the sought function in each subregion of the search zone consists of L values: $C = \{c_1, c_2, \ldots, c_L\}$ (below, this set is referred to as a master set). Let $a = (a_k, k = 1, 2, \ldots, K)$, where $a_k \in C$, be the vector whose components are values of the function in all subregions of the search zone (below, it is called an image).

From the statistical standpoint, measurement results, noise, and model parameters (the values of a discrete function) are considered as random variables. We should note that neither the image of the sought function nor the noise is known a priori but, if a certain vector is solution (1) at $y = y_0$, the misfit $e = y_0 - h(a_0)$ must be equal to w. Thus, we have

$$p(y_0/a_0) = p(e) = p(w). \tag{2}$$

In other words, if $a = a_0$, the event $y_0 = h(a_0) + w$ is equivalent to the event $e = w$.

The goal of the Bayesian approach is the calculation of the a posteriori PDF, i.e., the calculation of the corresponding conditional probabilities with given data and a noise level. The standard calculation of conditional probabilities for each image a defined on a set A and for a data set y provides an a posteriori PDF. According to the Bayes theorem, we have

$$p\sigma = a/Y = y = \frac{f(y/a)q(a)}{\sum_{b \in A} f(y/b)q(b)}, \tag{3}$$

where $q(a)$ is the a priori probability of the image a and $f(y/a)$ is the conditional probability that the data set assumes the value $y = (y_{ij}; i = 1, 2, \ldots, I; j = 1, 2, \ldots, J)$ at given values of the electrical conductivity. This probability depends on $a = a_k$; $k = 1, 2, \ldots, K$ and can be directly calculated using the formula

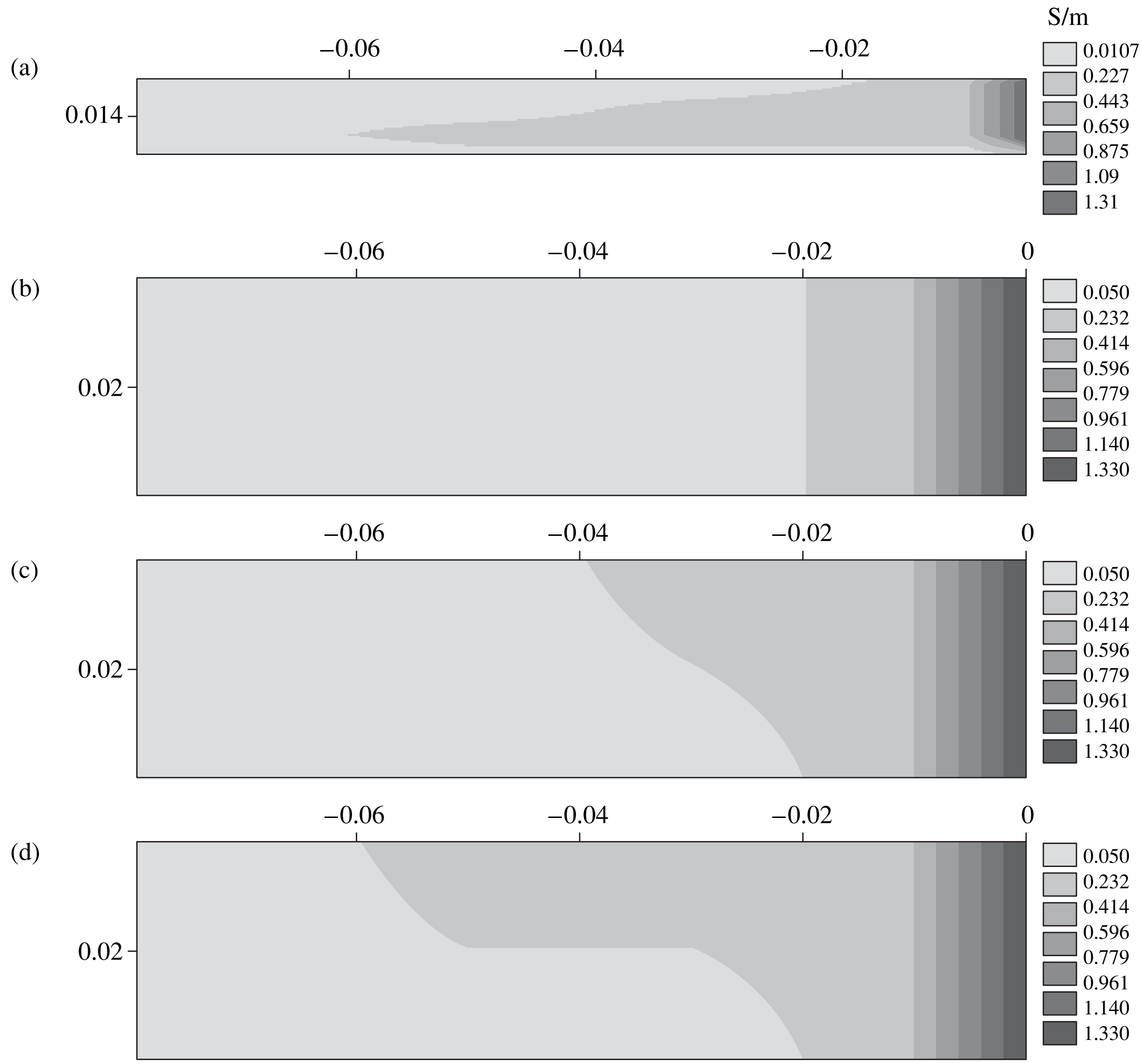

Fig. 2. Vertical (a) and horizontal (b–d) cross sections of the e posteriori distribution of the electrical conductivity in the search zone.

$$f(y/a) = \prod_{i=1}^{I}\prod_{j=1}^{J} p_{ij}(y_{ij} - h(\mathbf{E}(M_i, \omega_j, a), \mathbf{H}(M_i, \omega_j, a))), \tag{4}$$

where p_{ij} is the probability density of the noise w_{ij}. If the probability densities p_{ij} are Gaussian with a zero mean and variances δ_{ij}, formula (4) takes the form

$$f(y/a) = Z\exp\left(-\sum_{i,j}\frac{(y_{ij} - h(\mathbf{E}(M_i, \omega_j, a), \mathbf{H}(M_i, \omega_j, a)))^2}{2(\delta_{ij})^2}\right), \tag{5}$$

where Z is a normalizing constant. For convenience of calculations, we consider below that the function $f(y/a)$ is determined by formula (5), although it is important to note that this imposes no limitations on the applicability domain of the approach under consideration.

Let $\sigma_k^{(n)}$ $(k = 1, 2, \ldots, K)$ be function values after n iterations in the homogeneous subregions in the search zone. Scanning of a subregion $k_{(n)}$ at the $(n + 1)$th iteration changes the image of the conductivity due to the replacement of the conductivity value only in this subregion, and the new value is taken from the master set by random selection in accordance with the corresponding probability

$$p(\sigma_{k(n)}^{(n+1)} = c_l) = \frac{f(y/a(\sigma^{(n)}, k(n), c_l))q(a(\sigma^{(n)}, k(n), c_l))}{\sum_{l=1}^{L} f(y/a(\sigma^{(n)}, k(n), c_l))q(a(\sigma^{(n)}, k(n), c_l))}, \tag{6}$$

where $a(\sigma^{(n)}, k, c_l)$ denotes the image which is equal to c_l in the kth subregion and to $\sigma^{(n)}$ in all remaining subregions of the search zone. In order to find $p(\sigma_{k(n)}^{(n+1)} = c_l)$, it is necessary to calculate $f(y/a(\sigma^{(n)}, k(n), c_l))q(a(\sigma^{(n)}, k(n), c_l))$ L times. Thus, the total number of solutions of the forward problem per one iteration of the outer loop can be reduced to $L \cdot K$ [Spichak et al., 1999].

The sequence of images $\sigma^{(n)}$ ($n = 1, 2, \ldots$) forms a random process that is a Markovian chain in the space of all possible images. It can be proved that, for each subregion of the search zone, the sequence of average conditional probabilities converges to the corresponding marginal probability:

$$po_k(c_l) = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} p_k^n(c_l). \qquad (7)$$

After the probabilities are found, a posteriori values of the electrical conductivity can be estimated in each subregion of the search zone:

$$\sigma_k = \sum_{l=1}^{L} c_l po_k(c_l)(k = 1, 2, \ldots, K). \qquad (8)$$

Thus, the data inversion reduces to the search for the a posteriori distribution of the sought function by successively solving the forward problem for values chosen from the master set with regard for their a priori probabilities specified by a geophysicist-interpreter on the basis of the available geological and geophysical information, results of previous interpretations, and personal experience [Spichak et al., 1999].

ESTIMATION OF THE AMT DATA RESOLUTION

In order to estimate the resolution of AMT data in relation to the detection of salinization of coastal reservoirs by means of the model shown in Fig. 1, the model MT data were inverted with the help of the INVERS-3D program implementing the Bayesian approach. Because the constructed model of the electrical conductivity has a vertical symmetry plane (Fig. 1), computations were performed for only half of the model region. In this procedure, the search region was divided into 32 subregions; each subregion assumed, with equal probability, the constant conductivity values $\sigma_{FW} = 0.05$ S/m or $\sigma_{SW} = 0.4$ S/m. The forward problem was solved 12 times at the inner loop in each subregion (for each of the two a priori values of the electrical conductivity at three periods of the field with two polarizations). The inversion results were obtained after 37 iterations of the outer loop.

Figure 2 presents a posteriori conductivity distributions in a central vertical cross section (a) and in horizontal cross sections (b–d) at respective depths of 2, 4, and 6 m. In the lowest layer of saline water (6–8 m, see Fig. 2), the a posteriori conductivity did not exceed the average a priori value amounting to 0.225 S/m. The three upper layers were "detected" and delineated quite reliably. In particular, Fig. 2 clearly shows the boundaries of the saline zone, in both the vertical and horizontal projections.

Thus, we arrive at the following conclusions.

(1) The AMT sounding method can be successfully applied to solving problems related to the monitoring of variations (e.g., salinization and contamination) in a shallow crustal layer (a few tens of meters) if they change the electrical conductivity of the layer.

(2) The MT data inversion method based on the Bayesian statistic is a tool that can be effectively used for solving inverse problems of geoelectrics requiring formal inclusion of expert estimates.

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