SOME ISSUES OF THE THEORIES OF THIN BODIES

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Contents

For a long time, scientists have been paying attention to gradient theories. Among these theories, a special place is occupied by second-gradient theories with respect to the displacement vector and the strain tensor. They have been researched by Jaramillo, Mindlin, Toupin and etc.
Today scientist like Aifantis, Askes, F. dell'Isola, C. Polizzotto, H.Altenbach, W.Muller, P.Seppech and etc., and the authors of this report continue to develop gradient theories.

In particular, the authors, based on three-dimensional gradient theories, construct the corresponding gradient theories of thin bodies. For this we use the method of orthogonal polynomials. A thin body is a three-dimensional body, one dimension of which is smaller than the others, or it is a three-dimensional body, two dimensions of which are smaller than the third.

The presentation will mainly focus on the generalized second-gradient theory with respect to the strain tensor and the velocity vector, for which the constitutive relations and differential equations of motion and equilibrium are obtained. Then, based on these equations, we get the corresponding equations of the second-gradient theory of prismatic thin bodies with one small size.

$$\Phi(\underset{\sim}{\boldsymbol{\gamma}},\underset{\sim}{\boldsymbol{\varkappa}}) = \frac{1}{2}\big(A^{ijkl}\gamma_{ij}\gamma_{kl} + 2B^{ijkl}\gamma_{ij}\varkappa_{kl} + D^{ijkl}\varkappa_{ij}\varkappa_{kl}\big) = $$
$$= \frac{1}{2}\big(\underset{\sim}{\boldsymbol{\gamma}}\overset{2}{\otimes}\underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}} + 2\underset{\sim}{\boldsymbol{\gamma}}\overset{2}{\otimes}\underset{\approx}{\mathbf{B}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varkappa}} + \underset{\sim}{\boldsymbol{\varkappa}}\overset{2}{\otimes}\underset{\approx}{\mathbf{D}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varkappa}}\big).$$

$$\underset{\sim}{\mathbf{P}} = \frac{\partial\Phi}{\partial\underset{\sim}{\boldsymbol{\gamma}}} = \underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}} + \underset{\approx}{\mathbf{B}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varkappa}}, \quad \underset{\sim}{\boldsymbol{\mu}} = \frac{\partial\Phi}{\partial\underset{\sim}{\boldsymbol{\varkappa}}} = \underset{\approx}{\mathbf{C}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}} + \underset{\approx}{\mathbf{D}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varkappa}}, \quad \underset{\approx}{\mathbf{B}}^T = \underset{\approx}{\mathbf{C}},$$
$$P^{ij} = A^{ijkl}\gamma_{kl} + B^{ijkl}\varkappa_{kl}, \quad \mu^{ij} = C^{ijkl}\gamma_{kl} + D^{ijkl}\varkappa_{kl}, \quad C_{ijkl} = B_{klij},$$

Here "T"hereinafter means the operation of transposition
$\underset{\sim}{\mathbf{P}}$ is a stress tensor, $\underset{\sim}{\boldsymbol{\mu}}$ is a moment stress tensor
$\overset{n}{\otimes}$ is an n-inner product
$\Phi(\underset{\sim}{\boldsymbol{\gamma}},\underset{\sim}{\boldsymbol{\varkappa}})$ is the energy density
$\underset{\sim}{\boldsymbol{\gamma}} = \nabla\mathbf{u} - \underset{\approx}{\mathbf{C}}\cdot\boldsymbol{\varphi}$ is a strain tensor,

$\underset{\sim}{\boldsymbol{\varkappa}} = \nabla\boldsymbol{\varphi}$ is a bending-torsion tensor
$\underset{\approx}{\mathbf{A}}$, $\underset{\approx}{\mathbf{B}}$, $\underset{\approx}{\mathbf{C}}$, $\underset{\approx}{\mathbf{D}}$ are the fourth rank elastic modulus tensors
$\underset{\approx}{\tilde{\mathbf{C}}}$ is a third-rank discriminant tensor

$\mathbf{u}$ is the vector of displacements, $\boldsymbol{\varphi}$ is the vector of rotations

$$2\Phi(\underset{\sim}{\boldsymbol{\gamma}},\underset{\sim}{\boldsymbol{\varkappa}}) = \underset{\sim}{\mathbb{X}}^T \overset{2}{\otimes} \underset{\approx}{\mathbb{M}} \overset{2}{\otimes} \underset{\sim}{\mathbb{X}}, \quad \underset{\sim}{\mathbb{Y}} = \underset{\approx}{\mathbb{M}} \overset{2}{\otimes} \underset{\sim}{\mathbb{X}} \quad \left[\begin{pmatrix} \underset{\sim}{\mathbf{P}} \\ \underset{\sim}{\boldsymbol{\mu}} \end{pmatrix} = \begin{pmatrix} \underset{\approx}{\mathbf{A}} & \underset{\approx}{\mathbf{B}} \\ \underset{\approx}{\mathbf{C}} & \underset{\approx}{\mathbf{D}} \end{pmatrix} \overset{2}{\otimes} \begin{pmatrix} \underset{\sim}{\boldsymbol{\gamma}} \\ \underset{\sim}{\boldsymbol{\varkappa}} \end{pmatrix} \right].$$

where we introduce

$$\underset{\sim}{\mathbb{X}} = \begin{pmatrix} \underset{\sim}{\boldsymbol{\gamma}} \\ \underset{\sim}{\boldsymbol{\varkappa}} \end{pmatrix} \left(\underset{\sim}{\mathbb{X}}^T = \begin{pmatrix} \underset{\sim}{\boldsymbol{\gamma}}, \underset{\sim}{\boldsymbol{\varkappa}} \end{pmatrix}\right), \quad \underset{\sim}{\mathbb{Y}} = \begin{pmatrix} \underset{\sim}{\mathbf{P}} \\ \underset{\sim}{\boldsymbol{\mu}} \end{pmatrix} \left(\underset{\sim}{\mathbb{Y}}^T = \begin{pmatrix} \underset{\sim}{\mathbf{P}}, \underset{\sim}{\boldsymbol{\mu}}, \end{pmatrix}\right), \quad \underset{\approx}{\mathbb{M}} = \begin{pmatrix} \underset{\approx}{\mathbf{A}} & \underset{\approx}{\mathbf{B}} \\ \underset{\approx}{\mathbf{C}} & \underset{\approx}{\mathbf{D}} \end{pmatrix}$$

Energy density and constitutive relations in non-isothermal processes

$$\underset{\sim}{\mathbf{P}} = \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} (\underset{\sim}{\boldsymbol{\gamma}} - \underset{\sim}{\mathbf{a}}\vartheta) + \underset{\approx}{\mathbf{B}} \overset{2}{\otimes} (\underset{\sim}{\boldsymbol{\varkappa}} - \underset{\sim}{\mathbf{d}}\vartheta), \quad \underset{\sim}{\boldsymbol{\mu}} = \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} (\underset{\sim}{\boldsymbol{\gamma}} - \underset{\sim}{\mathbf{a}}\vartheta) + \underset{\approx}{\mathbf{D}} \overset{2}{\otimes} (\underset{\sim}{\boldsymbol{\varkappa}} - \underset{\sim}{\mathbf{d}}\vartheta),$$

Here $\vartheta = T - T_0$ is the temperature difference
$\underset{\sim}{\mathbf{a}}, \underset{\sim}{\mathbf{d}}$ are thermal expansion tensors

$$\underset{\sim}{\mathbf{P}} = \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \nabla \mathbf{u} + \underset{\approx}{\mathbf{B}} \overset{2}{\otimes} \nabla \boldsymbol{\varphi} - \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \underset{\simeq}{\mathbf{C}} \cdot \boldsymbol{\varphi} - \underset{\sim}{\mathbf{b}}\vartheta,$$
$$\underset{\sim}{\boldsymbol{\mu}} = \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \nabla \mathbf{u} + \underset{\approx}{\mathbf{D}} \overset{2}{\otimes} \nabla \boldsymbol{\varphi} - \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \underset{\simeq}{\mathbf{C}} \cdot \boldsymbol{\varphi} - \underset{\sim}{\boldsymbol{\beta}}\vartheta,$$

where for thermomechanical property tensors we introduced

$$\underset{\sim}{\mathbf{b}} = \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \underset{\sim}{\mathbf{a}} + \underset{\approx}{\mathbf{B}} \overset{2}{\otimes} \underset{\sim}{\mathbf{d}}, \quad \underset{\sim}{\boldsymbol{\beta}} = \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \underset{\sim}{\mathbf{a}} + \underset{\approx}{\mathbf{D}} \overset{2}{\otimes} \underset{\sim}{\mathbf{d}}$$

Geremillo material

$$\underset{\sim}{\mathbf{P}} = \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}} + {}^5\underset{\sim}{\mathbb{B}} \overset{3}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}}, \quad \underset{\approx}{\boldsymbol{\mu}} = {}^5\underset{\sim}{\mathbb{B}}' \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}} + {}^6\underset{\sim}{\mathbb{D}} \overset{3}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}} \quad [\underset{\sim}{\boldsymbol{\varepsilon}} = 1/2(\nabla\mathbf{u} + \nabla\mathbf{u}^T),\ \underset{\approx}{\boldsymbol{\varkappa}} = \nabla\underset{\sim}{\boldsymbol{\varepsilon}}],$$

$$P^{ij} = A^{ijkl}\varepsilon_{kl} + B^{ijklm}\varkappa_{klm}, \quad \mu^{ijk} = B'^{ijklm}\varepsilon_{lm} + D^{ijklmn}\varkappa_{lmn}, \quad B^{ijklm} = B^{klmij}.$$

If the material has a center of symmetry, then ${}^5\underset{\sim}{\mathbb{B}} = 0$, ${}^5\underset{\sim}{\mathbb{B}}' = 0$ and

$$\underset{\sim}{\mathbf{P}} = \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}} \ (P^{ij} = A^{ijkl}\varepsilon_{kl}),$$

$$\underset{\approx}{\boldsymbol{\mu}} = {}^6\underset{\sim}{\mathbb{D}} \overset{3}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}} \ (\mu^{ijk} = D^{ijklmn}\varkappa_{lmn}).$$

Here ${}^5\underset{\sim}{\mathbb{B}}$, ${}^6\underset{\sim}{\mathbb{D}}$ are fifth and sixth rank material tensors, respectively
$\underset{\sim}{\boldsymbol{\varepsilon}}$ is a strain tensor
$\underset{\approx}{\boldsymbol{\mu}}$ is a couple stress tensor

Non-linear second-gradient type material

In this case, instead of $\underset{\sim}{\varepsilon}$ and $\underset{\approx}{\varkappa} = \nabla \underset{\sim}{\varepsilon}$ in the previous material, we consider

$$\underset{\sim}{\mathbf{G}} = \underset{\sim}{\mathbf{F}} \cdot \underset{\sim}{\mathbf{F}}^T, \quad \underset{\approx}{\varkappa} = \nabla \underset{\sim}{\mathbf{G}},$$

respectively, where

$\underset{\sim}{\mathbf{F}} = \overset{\circ}{\nabla} \mathbf{r}$ is the position (motion) gradient
$\underset{\sim}{\mathbf{G}}$ is the Green's (Cauchy–Green's) strain tensor
$\mathbf{r}$ is a position vector in actual configuration
$\overset{\circ}{\nabla}$ is a nabla-operator in reference configuration

Tupin's material (gradient material)

Elastic strain energy

$$\Phi=\Phi(\underset{\sim}{\boldsymbol{\gamma}},\underset{\approx}{\boldsymbol{\varkappa}})=\frac{1}{2}\big(\underset{\sim}{\boldsymbol{\gamma}}\overset{2}{\otimes}\underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}}+2\underset{\sim}{\boldsymbol{\gamma}}\overset{2}{\otimes}{}^{5}\underset{\sim}{\mathbb{B}}\overset{3}{\otimes}\underset{\approx}{\boldsymbol{\varkappa}}+\underset{\approx}{\boldsymbol{\varkappa}}\overset{3}{\otimes}{}^{6}\underset{\sim}{\mathbb{D}}\overset{3}{\otimes}\underset{\approx}{\boldsymbol{\varkappa}}\big),\ \underset{\sim}{\boldsymbol{\gamma}}=\nabla\mathbf{u},\ \underset{\approx}{\boldsymbol{\varkappa}}=\nabla\nabla\mathbf{u},$$

Constitutive relations

$$\underset{\sim}{\mathbf{P}}=\frac{\partial\Phi}{\partial\underset{\sim}{\boldsymbol{\gamma}}}=\underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}}+{}^{5}\underset{\sim}{\mathbb{B}}\overset{3}{\otimes}\underset{\approx}{\boldsymbol{\varkappa}},\ \ \underset{\simeq}{\boldsymbol{\mu}}=\frac{\partial\Phi}{\partial\underset{\approx}{\boldsymbol{\varkappa}}}={}^{5}\underset{\sim}{\mathbb{B}}'\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}}+{}^{6}\underset{\sim}{\mathbb{D}}\overset{3}{\otimes}\underset{\approx}{\boldsymbol{\varkappa}},$$

$$P^{ij}=A^{ijkl}\gamma_{kl}+B^{ijklm}\varkappa_{klm},\quad \mu^{ijk}=B'^{ijklm}\gamma_{lm}+D^{ijklmn}\varkappa_{lmn},$$

$$B'^{ijklm}=B^{lmijk},\quad \gamma_{ij}=\nabla_i u_j,\quad \varkappa_{ijk}=\nabla_i\gamma_{jk}.$$

$$\mathbb{X}=\begin{pmatrix}\underset{\sim}{\boldsymbol{\gamma}}\\ \underset{\approx}{\boldsymbol{\varkappa}}\end{pmatrix},\quad \mathbb{Y}=\begin{pmatrix}\underset{\sim}{\mathbf{P}}\\ \underset{\simeq}{\boldsymbol{\mu}}\end{pmatrix},\quad \mathbb{M}=\begin{pmatrix}\underset{\approx}{\mathbf{A}} & {}^{5}\underset{\sim}{\mathbb{B}}\\ {}^{5}\underset{\sim}{\mathbb{B}}' & {}^{6}\underset{\sim}{\mathbb{D}}\end{pmatrix},$$

$$2\Phi(\underset{\sim}{\boldsymbol{\gamma}},\underset{\approx}{\boldsymbol{\varkappa}}) = \mathbb{X}^T \, {}^2\!\otimes_3 \mathbb{M} \, {}^2\!\otimes_3 \, \mathbb{X},$$

$$\begin{pmatrix} \underset{\sim}{\mathbf{P}} \\ \underset{\approx}{\boldsymbol{\mu}} \end{pmatrix} = \begin{pmatrix} \underset{\approx}{\mathbf{A}} & {}^5\underset{\sim}{\mathbb{B}} \\ {}^5\underset{\sim}{\mathbb{B}}' & {}^6\underset{\sim}{\mathbb{D}} \end{pmatrix} {}^2\!\otimes_3 \begin{pmatrix} \underset{\sim}{\boldsymbol{\gamma}} \\ \underset{\approx}{\boldsymbol{\varkappa}} \end{pmatrix}$$

or $\boxed{\mathbb{Y} = \mathbb{M} \, {}^2\!\otimes_3 \, \mathbb{X}}$, where $\left({}^2\!\otimes_3 = \begin{pmatrix} \overset{2}{\otimes} & 0 \\ 0 & \overset{3}{\otimes} \end{pmatrix} \right)$.

If the material has a center of symmetry then ${}^5\underset{\sim}{\mathbb{B}} = 0$, ${}^5\underset{\sim}{\mathbb{B}}' = 0$ and

$$\underset{\sim}{\mathbf{P}} = \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\gamma}}, \quad \underset{\approx}{\boldsymbol{\mu}} = {}^6\underset{\sim}{\mathbb{D}} \overset{3}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}},$$

$$P^{ij} = A^{ijkl}\gamma_{kl}, \quad \mu^{ijk} = D^{ijklmn}\varkappa_{lmn}.$$

Mindlin material

$$\underset{\sim}{\mathbf{P}}=\frac{\partial\Phi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}}=\underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varepsilon}}+\underset{\approx}{\mathbf{B}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}}+{}^5\underset{\sim}{\mathbb{C}}\overset{3}{\otimes}\underset{\simeq}{\boldsymbol{\varkappa}}\ (\underset{\sim}{\boldsymbol{\varepsilon}}=\frac{1}{2}(\nabla_x\mathbf{u}+\nabla_x\mathbf{u}^T),\ \underset{\approx}{\mathbf{A}}=\underset{\approx}{\mathbf{A}}^T,\ \underset{\approx}{\mathbf{B}}=\underset{\approx}{\mathbf{B}}^T),$$

$$\underset{\sim}{\mathbf{Q}}=\frac{\partial\Phi}{\partial\underset{\sim}{\boldsymbol{\gamma}}}=\underset{\approx}{\mathbf{B}}^T\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varepsilon}}+\underset{\approx}{\mathbf{D}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}}+{}^5\underset{\sim}{\mathbb{F}}\overset{3}{\otimes}\underset{\simeq}{\boldsymbol{\varkappa}}\ (\underset{\sim}{\boldsymbol{\gamma}}=\nabla_x\mathbf{u}-\underset{\sim}{\boldsymbol{\psi}},\ \underset{\sim}{\boldsymbol{\psi}}=\nabla_\xi\mathbf{v},\ \underset{\approx}{\mathbf{D}}=\underset{\approx}{\mathbf{D}}^T),$$

$$\underset{\simeq}{\boldsymbol{\mu}}=\frac{\partial\Phi}{\partial\underset{\simeq}{\boldsymbol{\varkappa}}}={}^5\underset{\sim}{\mathbb{C}}'\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varepsilon}}+{}^5\underset{\sim}{\mathbb{F}}'\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}}+{}^6\underset{\sim}{\mathbb{H}}\overset{3}{\otimes}\underset{\simeq}{\boldsymbol{\varkappa}}\ (\underset{\simeq}{\boldsymbol{\varkappa}}=\nabla_x\nabla_\xi\mathbf{v}=\nabla_x\underset{\sim}{\boldsymbol{\psi}},\ {}^6\underset{\sim}{\mathbb{H}}={}^6\underset{\sim}{\mathbb{H}}^T),$$

where

$$\Phi=\frac{1}{2}\underset{\sim}{\boldsymbol{\varepsilon}}\overset{2}{\otimes}\underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varepsilon}}+\frac{1}{2}\underset{\sim}{\boldsymbol{\gamma}}\overset{2}{\otimes}\underset{\approx}{\mathbf{D}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}}+\frac{1}{2}\underset{\simeq}{\boldsymbol{\varkappa}}\overset{3}{\otimes}{}^6\underset{\sim}{\mathbb{H}}\overset{3}{\otimes}\underset{\simeq}{\boldsymbol{\varkappa}}+\underset{\sim}{\boldsymbol{\varepsilon}}\overset{2}{\otimes}\underset{\approx}{\mathbf{B}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}}+\underset{\sim}{\boldsymbol{\varepsilon}}\overset{2}{\otimes}{}^5\underset{\sim}{\mathbb{C}}\overset{3}{\otimes}\underset{\simeq}{\boldsymbol{\varkappa}}+\underset{\sim}{\boldsymbol{\gamma}}\overset{2}{\otimes}{}^5\underset{\sim}{\mathbb{F}}\overset{3}{\otimes}\underset{\simeq}{\boldsymbol{\varkappa}},$$

$\mathbf{x}$ is a macroradius vector, $\boldsymbol{\xi}$ is a microradius vector, $\mathbf{u}$ is a macrodisplacement vector, $\mathbf{v}$ is a microdisplacement vector,
$\underset{\sim}{\boldsymbol{\varepsilon}}$ is a macrostrain tensor, $\underset{\sim}{\boldsymbol{\gamma}}$ is a relative distortion tensor, $\underset{\sim}{\boldsymbol{\psi}}=\nabla_{\boldsymbol{\xi}}\mathbf{v}$ is a microdistortion tensor, $\underset{\simeq}{\boldsymbol{\varkappa}}=\nabla_{\mathbf{x}}\underset{\sim}{\boldsymbol{\psi}}$ is a macrogradient of the microdistortion tensor

(these tensors are not depend on microcoordinates),
$\underset{\sim}{\mathbf{P}}$ is a symmetric stress tensor, $\underset{\sim}{\mathbf{Q}}$ is an asymmetric relative stress tensor,
$\underset{\simeq}{\boldsymbol{\mu}}$ is a couple stress tensor,

Φ is the strain energy per unit of volume.

$$P^{ij}=A^{ijkl}\varepsilon_{kl}+B^{ijkl}\gamma_{kl}+C^{ijklm}\varkappa_{klm} \qquad (\varepsilon_{ij}=\frac{1}{2}(\nabla_i u_j+\nabla_j u_i),\ B^{ijkl}=B^{klij}),$$
$$Q^{ij}=B'^{ijkl}\varepsilon_{kl}+D^{ijkl}\gamma_{kl}+F^{ijklm}\varkappa_{klm} \qquad (\gamma_{ij}=\nabla_i u_j-\psi_{ij},\ C^{ijklm}=C^{klmij}),$$
$$\mu^{ijk}=C'^{ijklm}\varepsilon_{lm}+F'^{ijklm}\gamma_{lm}+H^{ijklmn}\varkappa_{lmn} \qquad (\varkappa_{ijk}=\nabla_i\psi_{jk},\ F^{ijklm}=F^{klmij})$$

If the material has a center of symmetry, then ${}^5\underset{\sim}{\mathbb{C}}=0$, ${}^5\underset{\sim}{\mathbb{F}}=0$ and

$$\underset{\sim}{\mathbf{P}}=\frac{\partial\Phi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}}=\underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varepsilon}}+\underset{\approx}{\mathbf{B}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}},\quad \underset{\sim}{\mathbf{Q}}=\frac{\partial\Phi}{\partial\underset{\sim}{\boldsymbol{\gamma}}}=\underset{\approx}{\mathbf{B}}^T\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varepsilon}}+\underset{\approx}{\mathbf{D}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}},\quad \underset{\simeq}{\boldsymbol{\mu}}={}^6\underset{\sim}{\mathbb{H}}\overset{3}{\otimes}\underset{\simeq}{\boldsymbol{\varkappa}},$$

$$\Phi=\frac{1}{2}\underset{\sim}{\boldsymbol{\varepsilon}}\overset{2}{\otimes}\underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varepsilon}}+\frac{1}{2}\underset{\sim}{\boldsymbol{\gamma}}\overset{2}{\otimes}\underset{\approx}{\mathbf{D}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}}+\frac{1}{2}\underset{\simeq}{\boldsymbol{\varkappa}}\overset{3}{\otimes}{}^6\underset{\sim}{\mathbb{H}}\overset{3}{\otimes}\underset{\simeq}{\boldsymbol{\varkappa}}+\underset{\sim}{\boldsymbol{\varepsilon}}\overset{2}{\otimes}\underset{\approx}{\mathbf{B}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\gamma}}.$$

$$\underset{\sim}{\mathbb{X}}=\begin{pmatrix}\underset{\sim}{\boldsymbol{\varepsilon}}\\ \underset{\sim}{\boldsymbol{\gamma}}\end{pmatrix},\ \underset{\sim}{\mathbb{Y}}=\begin{pmatrix}\underset{\sim}{\mathbf{P}}\\ \underset{\sim}{\mathbf{Q}}\end{pmatrix},\ \underset{\approx}{\mathbb{M}}=\begin{pmatrix}\underset{\approx}{\mathbf{A}} & \underset{\approx}{\mathbf{B}}\\ \underset{\approx}{\mathbf{B}}^T & \underset{\approx}{\mathbf{D}}\end{pmatrix};\ \begin{pmatrix}\underset{\approx}{\mathbf{A}} & \underset{\approx}{\mathbf{B}}\\ \underset{\approx}{\mathbf{B}}^T & \underset{\approx}{\mathbf{D}}\end{pmatrix}\overset{2}{\otimes}\begin{pmatrix}\underset{\sim}{\boldsymbol{\varepsilon}}\\ \underset{\sim}{\boldsymbol{\gamma}}\end{pmatrix}=\begin{pmatrix}\underset{\sim}{\mathbf{P}}\\ \underset{\sim}{\mathbf{Q}}\end{pmatrix}$$

or $\underset{\approx}{\mathbb{M}}\overset{2}{\otimes}\underset{\sim}{\mathbb{X}}=\underset{\sim}{\mathbb{Y}}$.

$$\mathbb{M}=\begin{pmatrix}\underset{\approx}{\mathbf{A}} & \underset{\approx}{\mathbf{B}} & {}^5\underset{\sim}{\mathbb{C}}\\ \underset{\approx}{\mathbf{B}}^T & \underset{\approx}{\mathbf{D}} & {}^5\underset{\sim}{\mathbb{F}}\\ {}^5\underset{\sim}{\mathbb{C}}' & {}^5\underset{\sim}{\mathbb{F}}' & {}^6\underset{\sim}{\mathbb{H}}\end{pmatrix},\ \mathbb{X}=\begin{pmatrix}\underset{\sim}{\boldsymbol{\varepsilon}}\\ \underset{\sim}{\boldsymbol{\gamma}}\\ \underset{\simeq}{\boldsymbol{\varkappa}}\end{pmatrix},\ \mathbb{Y}=\begin{pmatrix}\underset{\sim}{\mathbf{P}}\\ \underset{\sim}{\mathbf{Q}}\\ \underset{\simeq}{\boldsymbol{\mu}}\end{pmatrix}.$$

Mindlin and Eshel material

Elastic strain energy

$$\Phi=\Phi(\underset{\sim}{\boldsymbol{\varepsilon}},\underset{\approx}{\boldsymbol{\varkappa}})=\frac{1}{2}\left(\underset{\sim}{\boldsymbol{\varepsilon}}\overset{2}{\otimes}\underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varepsilon}}+\underset{\approx}{\boldsymbol{\varkappa}}\overset{3}{\otimes}{}^{6}\underset{\sim}{\mathbf{D}}\overset{3}{\otimes}\underset{\approx}{\boldsymbol{\varkappa}}\right), \quad \underset{\sim}{\boldsymbol{\varepsilon}}=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T), \quad \underset{\approx}{\boldsymbol{\varkappa}}=\nabla\nabla\mathbf{u}.$$

Constitutive relations

$$\underset{\sim}{\mathbf{P}}=\frac{\partial\Phi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}}=\underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\underset{\sim}{\boldsymbol{\varepsilon}}, \quad \underset{\simeq}{\boldsymbol{\mu}}=\frac{\partial\Phi}{\partial\underset{\approx}{\boldsymbol{\varkappa}}}={}^{6}\underset{\sim}{\mathbf{D}}\overset{3}{\otimes}\underset{\approx}{\boldsymbol{\varkappa}},$$

$$P^{ij}=A^{ijkl}\varepsilon_{kl}, \quad \mu^{ijk}=D^{ijklmn}\varkappa_{lmn},$$

$$\varepsilon_{ij}=\frac{1}{2}(\nabla_i u_j+\nabla_j u_i), \quad \varkappa_{ijk}=\varkappa_{jik}=\nabla_i\nabla_j u_k.$$

$$A_{ijkl}=A_{jikl}=A_{ijlk}=A_{klij}, \quad D_{ijklmn}=D_{jiklmn}=D_{ijkmln}=D_{lmnijk}$$

The arbitrary isotropic tensors $\underset{\approx}{\mathbf{A}}$ and ${}^6\underset{\sim}{\mathbf{D}}$ have representations

$$\begin{aligned}
&\underset{\approx}{\mathbf{A}} = a_1 \underset{\approx}{\mathbf{C}}_{(1)} + a_2 \underset{\approx}{\mathbf{C}}_{(2)} + a_3 \underset{\approx}{\mathbf{C}}_{(3)},\\
&{}^6\underset{\sim}{\mathbf{D}} = b_1 \underset{\sim}{\mathbf{E}}\underset{\sim}{\mathbf{E}}\underset{\sim}{\mathbf{E}} + b_2 \underset{\sim}{\mathbf{E}}\underset{\approx}{\mathbf{C}}_{(2)} + b_3 \underset{\sim}{\mathbf{E}}\underset{\approx}{\mathbf{C}}_{(3)} + b_4 \underset{\approx}{\mathbf{C}}_{(2)}\underset{\sim}{\mathbf{E}} + b_5 \mathbf{r}_i\mathbf{r}_j\mathbf{r}^i\mathbf{r}_k\mathbf{r}^j\mathbf{r}^k + b_6 \mathbf{r}_i\mathbf{r}_j\mathbf{r}^i\underset{\sim}{\mathbf{E}}\mathbf{r}^j +\\
&+ b_7 \underset{\approx}{\mathbf{C}}_{(3)}\underset{\sim}{\mathbf{E}} + b_8 \mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k + b_9 \mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}^i\mathbf{r}^k\mathbf{r}^j + b_{10} \mathbf{r}_i\underset{\sim}{\mathbf{E}}\mathbf{r}_j\mathbf{r}^i\mathbf{r}^j + b_{11} \mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}^j\mathbf{r}^i\mathbf{r}^k +\\
&+ b_{12} \mathbf{r}_i\mathbf{r}_j\underset{\sim}{\mathbf{E}}\mathbf{r}^i\mathbf{r}^j + b_{13} \mathbf{r}_i\underset{\approx}{\mathbf{C}}_{(1)}\mathbf{r}^i + b_{14} \mathbf{r}_i\underset{\approx}{\mathbf{C}}_{(2)}\mathbf{r}^i + b_{15} \mathbf{r}_i\underset{\approx}{\mathbf{C}}_{(3)}\mathbf{r}^i.
\end{aligned}$$

Here $\underset{\sim}{\mathbf{E}}$ is the second rank unit tensor
$\underset{\approx}{\mathbf{C}}_{(1)}$, $\underset{\approx}{\mathbf{C}}_{(2)}$ and $\underset{\approx}{\mathbf{C}}_{(3)}$ are isotropic tensors of the fourth rank
$\mathbf{r}_i, \mathbf{r}^i$ are covariant and contravariant basis vectors, respectively

Taking into account the symmetries of the components of the tensors $\underset{\approx}{\mathbf{A}}$ and ${}^6\underset{\sim}{\mathbf{D}}$,

we obtain
$$a_2 = a_3, \quad d_1 \equiv b_1 = b_2 = b_6 = b_{13}, \quad d_2 \equiv b_4 = b_5 = b_7 = b_{10},$$
$$d_3 \equiv b_3, \quad d_4 \equiv b_8 = b_{11}, \quad d_5 \equiv b_9 = b_{12} = b_{14} = b_{15},$$

$$\underset{\approx}{\mathbf{A}} = a_1 \underset{\approx}{\mathbf{C}}_{(1)} + a_2(\underset{\approx}{\mathbf{C}}_{(2)} + \underset{\approx}{\mathbf{C}}_{(3)}),$$
$${}^6\underset{\sim}{\mathbf{D}} = d_1(\underset{\sim}{\mathbf{E}}\underset{\sim}{\mathbf{E}}\underset{\sim}{\mathbf{E}} + \underset{\sim}{\mathbf{E}}\underset{\approx}{\mathbf{C}}_{(2)} + \mathbf{r}_i\mathbf{r}_j\mathbf{r}^i\underset{\sim}{\mathbf{E}}\mathbf{r}^j + \mathbf{r}_i\underset{\approx}{\mathbf{C}}_{(1)}\mathbf{r}^i) + d_2(2\underset{\approx}{\mathbf{E}}\underset{\sim}{\mathbf{E}} + \mathbf{r}_i\mathbf{r}_j\mathbf{r}^i\mathbf{r}_k\mathbf{r}^j\mathbf{r}^k + \mathbf{r}_i\underset{\sim}{\mathbf{E}}\mathbf{r}_j\mathbf{r}^i\mathbf{r}^j)+$$
$$+d_3\underset{\sim}{\mathbf{E}}\underset{\approx}{\mathbf{C}}_{(3)} + d_4({}^6\underset{\sim}{\mathbf{E}} + \mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}^j\mathbf{r}^i\mathbf{r}^k) + d_5(2\mathbf{r}_i\underset{\approx}{\mathbf{E}}\mathbf{r}^i + \mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}^i\mathbf{r}^k\mathbf{r}^j + \mathbf{r}_i\mathbf{r}_j\underset{\sim}{\mathbf{E}}\mathbf{r}^i\mathbf{r}^j),$$

where $\underset{\approx}{\mathbf{E}}$ is the fourth rank unit tensor with respect to the inner 2-product operation
${}^6\underset{\sim}{\mathbf{E}} = \mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k$ is the sixth rank unit tensor with respect to the inner 3-product operation
If the components of the tensor ${}^6\underset{\sim}{\mathbf{D}}$ have the symmetry $D_{ijklmn} = D_{ijlkmn}$, then $d_1 = d_3 = d_5, d_2 = d_4$, and the tensor ${}^6\underset{\sim}{\mathbf{D}}$ is represented as

$${}^6\underset{\sim}{\mathbf{D}} = d_1[\underset{\sim}{\mathbf{E}}(\underset{\approx}{\mathbf{C}}_{(1)} + \underset{\approx}{\mathbf{C}}_{(2)} + \underset{\approx}{\mathbf{C}}_{(3)}) + \mathbf{r}_i(\underset{\approx}{\mathbf{C}}_{(1)} + \underset{\approx}{\mathbf{C}}_{(2)} + \underset{\approx}{\mathbf{C}}_{(3)})\mathbf{r}^i + \mathbf{r}_i\mathbf{r}_j\mathbf{r}^i\underset{\sim}{\mathbf{E}}\mathbf{r}^j + \mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}^i\mathbf{r}^k\mathbf{r}^j +$$
$$+\mathbf{r}_i\mathbf{r}_j\underset{\sim}{\mathbf{E}}\mathbf{r}^i\mathbf{r}^j] + d_2[{}^6\underset{\sim}{\mathbf{E}} + (\underset{\approx}{\mathbf{C}}_{(2)} + \underset{\approx}{\mathbf{C}}_{(3)})\underset{\sim}{\mathbf{E}} + \mathbf{r}_i\mathbf{r}_j\mathbf{r}^i\mathbf{r}_k\mathbf{r}^j\mathbf{r}^k + \mathbf{r}_i\underset{\sim}{\mathbf{E}}\mathbf{r}_j\mathbf{r}^i\mathbf{r}^j + \mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}^j\mathbf{r}^i\mathbf{r}^k].$$

A Second Strain Gradient Elasticity Theory with Second Velocity Gradient Inertia

The elastic strain energy and kinetic energy are represented as

$$\Phi = \Phi(\underset{\sim}{\boldsymbol{\varepsilon}}, \nabla\underset{\sim}{\boldsymbol{\varepsilon}}, \nabla\nabla\underset{\sim}{\boldsymbol{\varepsilon}}), \quad K = K(\mathbf{v}, \nabla\mathbf{v}, \nabla\nabla\mathbf{v})$$

where $\underset{\sim}{\boldsymbol{\varepsilon}}$ is the strain tensor
$\mathbf{v}$ is the velocity vector
Expanding the functions $\Phi(\underset{\sim}{\boldsymbol{\varepsilon}}, \nabla\underset{\sim}{\boldsymbol{\varepsilon}}, \nabla\nabla\underset{\sim}{\boldsymbol{\varepsilon}})$ and $K(\mathbf{v}, \nabla\mathbf{v}, \nabla\nabla\mathbf{v})$ to the Maclaurin series, we will have

$$\Phi = \Phi(\underset{\sim}{\boldsymbol{\varepsilon}}, \nabla\underset{\sim}{\boldsymbol{\varepsilon}}, \nabla\nabla\underset{\sim}{\boldsymbol{\varepsilon}}) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\underset{\sim}{\boldsymbol{\varepsilon}} \overset{2}{\otimes} \frac{\partial}{\partial \underset{\sim}{\boldsymbol{\varepsilon}}} + \nabla\underset{\sim}{\boldsymbol{\varepsilon}} \overset{3}{\otimes} \frac{\partial}{\partial \nabla\underset{\sim}{\boldsymbol{\varepsilon}}} + \nabla\nabla\underset{\sim}{\boldsymbol{\varepsilon}} \overset{4}{\otimes} \frac{\partial}{\partial \nabla\nabla\underset{\sim}{\boldsymbol{\varepsilon}}} \right)_0^{(k)} \Phi,$$

$$K = K(\mathbf{v}, \nabla\mathbf{v}, \nabla\nabla\mathbf{v}) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{v}} + \nabla\mathbf{v} \overset{2}{\otimes} \frac{\partial}{\partial \nabla\mathbf{v}} + \nabla\nabla\mathbf{v} \overset{3}{\otimes} \frac{\partial}{\partial \nabla\nabla\mathbf{v}} \right)_0^{(k)} K$$

Neglecting terms of the second order in the relations, we obtain

$$
\begin{aligned}
\Phi(\underset{\approx}{\boldsymbol{\varepsilon}},\underset{\simeq}{\boldsymbol{\gamma}},\underset{\approx}{\boldsymbol{\varkappa}}) = \Phi(\mathbf{0},\mathbf{0},\mathbf{0}) + \underset{\approx}{\boldsymbol{\varepsilon}} \overset{2}{\otimes} \left(\frac{\partial \Phi}{\partial \underset{\approx}{\boldsymbol{\varepsilon}}}\right)_{(0)} + \underset{\simeq}{\boldsymbol{\gamma}} \overset{3}{\otimes} \left(\frac{\partial \Phi}{\partial \underset{\simeq}{\boldsymbol{\gamma}}}\right)_{(0)} + \underset{\approx}{\boldsymbol{\varkappa}} \overset{4}{\otimes} \left(\frac{\partial \Phi}{\partial \underset{\approx}{\boldsymbol{\varkappa}}}\right)_{(0)} + \\
+\frac{1}{2}\Big[\underset{\approx}{\boldsymbol{\varepsilon}}\underset{\approx}{\boldsymbol{\varepsilon}} \overset{4}{\otimes} \left(\frac{\partial^2 \Phi}{\partial \underset{\approx}{\boldsymbol{\varepsilon}} \partial \underset{\approx}{\boldsymbol{\varepsilon}}}\right)_{(0)} + \underset{\approx}{\boldsymbol{\varepsilon}}\underset{\simeq}{\boldsymbol{\gamma}} \overset{5}{\otimes} \left(\frac{\partial^2 \Phi}{\partial \underset{\approx}{\boldsymbol{\varepsilon}} \partial \underset{\simeq}{\boldsymbol{\gamma}}}\right)_{(0)} + \underset{\approx}{\boldsymbol{\varepsilon}}\underset{\approx}{\boldsymbol{\varkappa}} \overset{6}{\otimes} \left(\frac{\partial^2 \Phi}{\partial \underset{\approx}{\boldsymbol{\varepsilon}} \partial \underset{\approx}{\boldsymbol{\varkappa}}}\right)_{(0)} + \\
+\underset{\simeq}{\boldsymbol{\gamma}}\underset{\approx}{\boldsymbol{\varepsilon}} \overset{5}{\otimes} \left(\frac{\partial^2 \Phi}{\partial \underset{\simeq}{\boldsymbol{\gamma}} \partial \underset{\approx}{\boldsymbol{\varepsilon}}}\right)_{(0)} + \underset{\simeq}{\boldsymbol{\gamma}}\underset{\simeq}{\boldsymbol{\gamma}} \overset{6}{\otimes} \left(\frac{\partial^2 \Phi}{\partial \underset{\simeq}{\boldsymbol{\gamma}} \partial \underset{\simeq}{\boldsymbol{\gamma}}}\right)_{(0)} + \underset{\simeq}{\boldsymbol{\gamma}}\underset{\approx}{\boldsymbol{\varkappa}} \overset{7}{\otimes} \left(\frac{\partial^2 \Phi}{\partial \underset{\simeq}{\boldsymbol{\gamma}} \partial \underset{\approx}{\boldsymbol{\varkappa}}}\right)_{(0)} + \\
+\underset{\approx}{\boldsymbol{\varkappa}}\underset{\approx}{\boldsymbol{\varepsilon}} \overset{6}{\otimes} \left(\frac{\partial^2 \Phi}{\partial \underset{\approx}{\boldsymbol{\varkappa}} \partial \underset{\approx}{\boldsymbol{\varepsilon}}}\right)_{(0)} + \underset{\approx}{\boldsymbol{\varkappa}}\underset{\simeq}{\boldsymbol{\gamma}} \overset{7}{\otimes} \left(\frac{\partial^2 \Phi}{\partial \underset{\approx}{\boldsymbol{\varkappa}} \partial \underset{\simeq}{\boldsymbol{\gamma}}}\right)_{(0)} + \underset{\approx}{\boldsymbol{\varkappa}}\underset{\approx}{\boldsymbol{\varkappa}} \overset{8}{\otimes} \left(\frac{\partial^2 \Phi}{\partial \underset{\approx}{\boldsymbol{\varkappa}} \partial \underset{\approx}{\boldsymbol{\varkappa}}}\right)_{(0)}\Big];
\end{aligned}
$$

$$
\begin{aligned}
K(\mathbf{v},\underset{\sim}{\mathbf{v}},\underset{\simeq}{\mathbf{v}}) = {} & K(\mathbf{0},\underset{\sim}{\mathbf{0}},\underset{\simeq}{\mathbf{0}}) + \mathbf{v}\cdot\left(\frac{\partial K}{\partial \mathbf{v}}\right)_{(0)} + \underset{\sim}{\mathbf{v}} \overset{2}{\otimes} \left(\frac{\partial K}{\partial \underset{\sim}{\mathbf{v}}}\right)_{(0)} + \underset{\simeq}{\mathbf{v}} \overset{3}{\otimes} \left(\frac{\partial K}{\partial \underset{\simeq}{\mathbf{v}}}\right)_{(0)} + \\
& + \frac{1}{2}\Bigg[\mathbf{v}\mathbf{v} \overset{2}{\otimes} \left(\frac{\partial^2 K}{\partial \mathbf{v}\partial \mathbf{v}}\right)_{(0)} + \mathbf{v}\underset{\sim}{\mathbf{v}} \overset{3}{\otimes} \left(\frac{\partial^2 K}{\partial \mathbf{v}\partial \underset{\sim}{\mathbf{v}}}\right)_{(0)} + \mathbf{v}\underset{\simeq}{\mathbf{v}} \overset{4}{\otimes} \left(\frac{\partial^2 K}{\partial \mathbf{v}\partial \underset{\simeq}{\mathbf{v}}}\right)_{(0)} + \\
& + \underset{\sim}{\mathbf{v}}\mathbf{v} \overset{3}{\otimes} \left(\frac{\partial^2 K}{\partial \underset{\sim}{\mathbf{v}}\partial \mathbf{v}}\right)_{(0)} + \underset{\sim}{\mathbf{v}}\underset{\sim}{\mathbf{v}} \overset{4}{\otimes} \left(\frac{\partial^2 K}{\partial \underset{\sim}{\mathbf{v}}\partial \underset{\sim}{\mathbf{v}}}\right)_{(0)} + \underset{\sim}{\mathbf{v}}\underset{\simeq}{\mathbf{v}} \overset{5}{\otimes} \left(\frac{\partial^2 K}{\partial \underset{\sim}{\mathbf{v}}\partial \underset{\simeq}{\mathbf{v}}}\right)_{(0)} + \\
& + \underset{\simeq}{\mathbf{v}}\mathbf{v} \overset{4}{\otimes} \left(\frac{\partial^2 K}{\partial \underset{\simeq}{\mathbf{v}}\partial \mathbf{v}}\right)_{(0)} + \underset{\simeq}{\mathbf{v}}\underset{\sim}{\mathbf{v}} \overset{5}{\otimes} \left(\frac{\partial^2 K}{\partial \underset{\simeq}{\mathbf{v}}\partial \underset{\sim}{\mathbf{v}}}\right)_{(0)} + \underset{\simeq}{\mathbf{v}}\underset{\simeq}{\mathbf{v}} \overset{6}{\otimes} \left(\frac{\partial^2 K}{\partial \underset{\simeq}{\mathbf{v}}\partial \underset{\simeq}{\mathbf{v}}}\right)_{(0)}\Bigg],
\end{aligned}
$$

where the following notations $\underset{\simeq}{\boldsymbol{\gamma}} = \nabla\underset{\sim}{\boldsymbol{\varepsilon}}$, $\underset{\approx}{\boldsymbol{\varkappa}} = \nabla\nabla\underset{\sim}{\boldsymbol{\varepsilon}}$, $\underset{\sim}{\mathbf{v}} = \nabla\mathbf{v}$ and $\underset{\simeq}{\mathbf{v}} = \nabla\nabla\mathbf{v}$ are introduced.

Note that for $\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\mathbf{0}}$, $\underset{\approx}{\boldsymbol{\gamma}} = \underset{\approx}{\mathbf{0}}$ and $\underset{\approx}{\boldsymbol{\varkappa}} = \underset{\approx}{\mathbf{0}}$ the elastic strain energy, and for $\mathbf{v} = \mathbf{0}$, $\underset{\sim}{\mathbf{v}} = \underset{\sim}{\mathbf{0}}$ and $\underset{\approx}{\mathbf{v}} = \underset{\approx}{\mathbf{0}}$ the kinetic energy, takes on a minimum value, which is zero. In this regard, the values of the first derivatives of the elastic strain energy and kinetic energy in their arguments are also equal to zero. So these two relations can be represented as

$$\Phi(\underset{\sim}{\boldsymbol{\varepsilon}},\underset{\approx}{\boldsymbol{\gamma}},\underset{\approx}{\boldsymbol{\varkappa}}) = \frac{1}{2}\Big({}^4\underset{\sim}{\mathbf{A}}^{(11)} \overset{4}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\boldsymbol{\varepsilon}} + 2\,{}^5\underset{\sim}{\mathbf{A}}^{(12)} \overset{5}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\approx}{\boldsymbol{\gamma}} + 2\,{}^6\underset{\sim}{\mathbf{A}}^{(13)} \overset{6}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\approx}{\boldsymbol{\varkappa}} + {}^6\underset{\sim}{\mathbf{A}}^{(22)} \overset{6}{\otimes} \underset{\approx}{\boldsymbol{\gamma}}\underset{\approx}{\boldsymbol{\gamma}} + 2\,{}^7\underset{\sim}{\mathbf{A}}^{(23)} \overset{7}{\otimes} \underset{\approx}{\boldsymbol{\gamma}}\underset{\approx}{\boldsymbol{\varkappa}} + {}^8\underset{\sim}{\mathbf{A}}^{(33)} \overset{8}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}}\underset{\approx}{\boldsymbol{\varkappa}}\Big);$$

$$K(\mathbf{v},\underset{\sim}{\mathbf{v}},\underset{\approx}{\mathbf{v}}) = \frac{1}{2}\Big({}^2\underset{\sim}{\mathbf{B}}^{(11)} \overset{2}{\otimes} \mathbf{v}\mathbf{v} + 2\,{}^3\underset{\sim}{\mathbf{B}}^{(12)} \overset{3}{\otimes} \mathbf{v}\underset{\sim}{\mathbf{v}} + 2\,{}^4\underset{\sim}{\mathbf{B}}^{(13)} \overset{4}{\otimes} \mathbf{v}\underset{\approx}{\mathbf{v}} + {}^4\underset{\sim}{\mathbf{B}}^{(22)} \overset{3}{\otimes} \underset{\sim}{\mathbf{v}}\underset{\sim}{\mathbf{v}} + 2\,{}^5\underset{\sim}{\mathbf{B}}^{(23)} \overset{5}{\otimes} \underset{\sim}{\mathbf{v}}\underset{\approx}{\mathbf{v}} + {}^6\underset{\sim}{\mathbf{B}}^{(33)} \overset{6}{\otimes} \underset{\approx}{\mathbf{v}}\underset{\approx}{\mathbf{v}}\Big).$$

Here we introduced the following notation

$$\overset{4}{\underset{\sim}{\mathbf{A}}}{}^{(11)} = \mathrm{A}^{(11)}_{ijkl}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l = \mathrm{A}^{(11)}_{klij}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l = \left(\frac{\partial^2\Phi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}\partial\underset{\sim}{\boldsymbol{\varepsilon}}}\right)_{(0)},$$

$$\overset{6}{\underset{\sim}{\mathbf{A}}}{}^{(22)} = A^{(22)}_{ijklmn}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m\mathbf{r}^n = A^{(22)}_{lmnijk}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m\mathbf{r}^n = \left(\frac{\partial^2\Phi}{\partial\underset{\simeq}{\boldsymbol{\gamma}}\partial\underset{\simeq}{\boldsymbol{\gamma}}}\right)_{(0)},$$

$$\overset{8}{\underset{\sim}{\mathbf{A}}}{}^{(33)} = A^{(33)}_{ijklmnpq}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m\mathbf{r}^n\mathbf{r}^p\mathbf{r}^q = A^{(33)}_{mnpqijkl}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m\mathbf{r}^n\mathbf{r}^p\mathbf{r}^q = \left(\frac{\partial^2\Phi}{\partial\underset{\approx}{\boldsymbol{\varkappa}}\partial\underset{\approx}{\boldsymbol{\varkappa}}}\right)_{(0)},$$

$$\overset{5}{\underset{\sim}{\mathbf{A}}}{}^{(12)} = A^{(12)}_{ijklm}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m = A^{(12)}_{klmij}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m = \frac{1}{2}\left[\left(\frac{\partial^2\Phi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}\partial\underset{\simeq}{\boldsymbol{\gamma}}}\right)_{(0)} + \left(\frac{\partial^2\Phi}{\partial\underset{\simeq}{\boldsymbol{\gamma}}\partial\underset{\sim}{\boldsymbol{\varepsilon}}}\right)^T_{(0)}\right],$$

$$\overset{6}{\underset{\sim}{\mathbf{A}}}{}^{(13)} = A^{(13)}_{ijklmn}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m\mathbf{r}^n = A^{(13)}_{klmnij}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m\mathbf{r}^n = \frac{1}{2}\left[\left(\frac{\partial^2\Phi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}\partial\underset{\approx}{\boldsymbol{\varkappa}}}\right)_{(0)} + \left(\frac{\partial^2\Phi}{\partial\underset{\approx}{\boldsymbol{\varkappa}}\partial\underset{\simeq}{\boldsymbol{\gamma}}}\right)^T_{(0)}\right]$$

$$\overset{7}{\underset{\sim}{\mathbf{A}}}{}^{(23)} = A^{(23)}_{ijklmnp}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m\mathbf{r}^n\mathbf{r}^p = A^{(23)}_{lmnpijk}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m\mathbf{r}^n\mathbf{r}^p =$$
$$= \frac{1}{2}\left[\left(\frac{\partial^2\Phi}{\partial\underset{\simeq}{\boldsymbol{\gamma}}\partial\underset{\approx}{\boldsymbol{\varkappa}}}\right)_{(0)} + \left(\frac{\partial^2\Phi}{\partial\underset{\approx}{\boldsymbol{\varkappa}}\partial\underset{\simeq}{\boldsymbol{\gamma}}}\right)^T_{(0)}\right];$$

$$
{}^{2}\underset{\sim}{\mathbf{B}}^{(11)} = B^{(11)}_{ij}\mathbf{r}^i\mathbf{r}^j = B^{(11)}_{ji}\mathbf{r}^i\mathbf{r}^j = \left(\frac{\partial^2 K}{\partial \mathbf{v}\partial \mathbf{v}}\right)_{(0)},
$$

$$
{}^{4}\underset{\sim}{\mathbf{B}}^{(22)} = B^{(22)}_{ijkl}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l = B^{(22)}_{klij}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l = \left(\frac{\partial^2 K}{\partial \underset{\sim}{\mathbf{v}}\partial \underset{\sim}{\mathbf{v}}}\right)_{(0)},
$$

$$
{}^{6}\underset{\sim}{\mathbf{B}}^{(33)} = B^{(33)}_{ijklmn}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m\mathbf{r}^n = B^{(33)}_{lmnijk}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m\mathbf{r}^n = \left(\frac{\partial^2 K}{\partial \underset{\approx}{\mathbf{v}}\partial \underset{\approx}{\mathbf{v}}}\right)_{(0)},
$$

$$
{}^{3}\underset{\sim}{\mathbf{B}}^{(12)} = B^{(12)}_{ijk}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k = B^{(12)}_{jki}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k = \frac{1}{2}\left[\left(\frac{\partial^2 K}{\partial \mathbf{v}\partial \underset{\sim}{\mathbf{v}}}\right)_{(0)} + \left(\frac{\partial^2 K}{\partial \underset{\sim}{\mathbf{v}}\partial \mathbf{v}}\right)^T_{(0)}\right],
$$

$$
{}^{4}\underset{\sim}{\mathbf{B}}^{(13)} = B^{(13)}_{ijkl}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l = B^{(13)}_{jkli}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l = \frac{1}{2}\left[\left(\frac{\partial^2 K}{\partial \mathbf{v}\partial \underset{\approx}{\mathbf{v}}}\right)_{(0)} + \left(\frac{\partial^2 K}{\partial \underset{\approx}{\mathbf{v}}\partial \mathbf{v}}\right)^T_{(0)}\right],
$$

$$
{}^{5}\underset{\sim}{\mathbf{B}}^{(23)} = B^{(23)}_{ijklm}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m = B^{(23)}_{klmij}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\mathbf{r}^m = \frac{1}{2}\left[\left(\frac{\partial^2 K}{\partial \underset{\sim}{\mathbf{v}}\partial \underset{\approx}{\mathbf{v}}}\right)_{(0)} + \left(\frac{\partial^2 K}{\partial \underset{\approx}{\mathbf{v}}\partial \underset{\sim}{\mathbf{v}}}\right)^T_{(0)}\right].
$$

The following symmetries take place

$$
\begin{gathered}
A^{(12)}_{ijklm}=A^{(12)}_{jiklm}=A^{(12)}_{ijkml},\quad A^{(13)}_{ijklmn}=A^{(13)}_{jiklmn}=A^{(13)}_{ijlkmn}=A^{(13)}_{ijklnm},\\
A^{(23)}_{ijklmnp}=A^{(23)}_{ikjlmnp}=A^{(23)}_{ijkmlnp}=A^{(23)}_{ijklmpn},\ A^{(11)}_{ijkl}=A^{(11)}_{jikl}=A^{(11)}_{ijlk}=A^{(11)}_{klij},\\
A^{(22)}_{ijklmn}=A^{(22)}_{ikjlmn}=A^{(22)}_{ijklnm}=A^{(22)}_{lmnijk};\quad B^{(13)}_{ijkl}=B^{(13)}_{ikjl},\quad B^{(23)}_{ijklm}=B^{(23)}_{ijlkm},\\
B^{(11)}_{ij}=B^{(11)}_{ji},\quad B^{(22)}_{ijkl}=B^{(22)}_{klij},\quad B^{(33)}_{ijklmn}=B^{(33)}_{ikjlmn}=B^{(33)}_{ijklnm}=B^{(33)}_{lmnijk}.
\end{gathered}
$$

Constitutive relations

$$\underset{\sim}{\mathbf{P}} = \frac{\partial \Phi}{\partial \underset{\sim}{\boldsymbol{\varepsilon}}} = {}^4\underset{\sim}{\mathbf{A}}^{(11)} \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}} + {}^5\underset{\sim}{\mathbf{A}}^{(12)} \overset{3}{\otimes} \underset{\approx}{\boldsymbol{\gamma}} + {}^6\underset{\sim}{\mathbf{A}}^{(13)} \overset{4}{\otimes} \underset{\sim}{\underset{\approx}{\boldsymbol{\varkappa}}},$$

$$\underset{\approx}{\mathbf{P}}^{(1)} = \frac{\partial \Phi}{\partial \nabla \underset{\sim}{\boldsymbol{\varepsilon}}} = ({}^5\underset{\sim}{\mathbf{A}}^{(12)})^T \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}} + {}^6\underset{\sim}{\mathbf{A}}^{(22)} \overset{3}{\otimes} \underset{\approx}{\boldsymbol{\gamma}} + {}^7\underset{\sim}{\mathbf{A}}^{(23)} \overset{4}{\otimes} \underset{\sim}{\underset{\approx}{\boldsymbol{\varkappa}}},$$

$$\underset{\sim}{\underset{\approx}{\mathbf{P}}}^{(2)} = \frac{\partial \Phi}{\partial \nabla\nabla \underset{\sim}{\boldsymbol{\varepsilon}}} = ({}^6\underset{\sim}{\mathbf{A}}^{(13)})^T \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}} + ({}^7\underset{\sim}{\mathbf{A}}^{(23)})^T \overset{3}{\otimes} \underset{\approx}{\boldsymbol{\gamma}} + {}^8\underset{\sim}{\mathbf{A}}^{(33)} \overset{4}{\otimes} \underset{\sim}{\underset{\approx}{\boldsymbol{\varkappa}}};$$

$$\boldsymbol{\pi} = \frac{\partial K}{\partial \mathbf{v}} = {}^2\underset{\sim}{\mathbf{B}}^{(11)} \overset{1}{\otimes} \mathbf{v} + {}^3\underset{\sim}{\mathbf{B}}^{(12)} \overset{2}{\otimes} \underset{\sim}{\mathbf{v}} + {}^4\underset{\sim}{\mathbf{B}}^{(13)} \overset{3}{\otimes} \underset{\approx}{\mathbf{v}},$$

$$\underset{\sim}{\boldsymbol{\pi}}^{(1)} = \frac{\partial K}{\partial \nabla \mathbf{v}} = ({}^3\underset{\sim}{\mathbf{B}}^{(12)})^T \overset{1}{\otimes} \mathbf{v} + {}^4\underset{\sim}{\mathbf{B}}^{(22)} \overset{2}{\otimes} \underset{\sim}{\mathbf{v}} + {}^5\underset{\sim}{\mathbf{B}}^{(23)} \overset{3}{\otimes} \underset{\approx}{\mathbf{v}},$$

$$\underset{\approx}{\boldsymbol{\pi}}^{(2)} = \frac{\partial K}{\partial \nabla\nabla \mathbf{v}} = ({}^4\underset{\sim}{\mathbf{B}}^{(13)})^T \overset{1}{\otimes} \mathbf{v} + ({}^5\underset{\sim}{\mathbf{B}}^{(23)})^T \overset{2}{\otimes} \underset{\sim}{\mathbf{v}} + {}^6\underset{\sim}{\mathbf{B}}^{(33)} \overset{3}{\otimes} \underset{\approx}{\mathbf{v}}.$$

Introducing tensor columns and tensor-block matrices the constitutive relations can be written in short form $\boxed{\underset{\sim}{\mathbb{Y}} = \mathbb{M} \odot \mathbb{X}}$, $\boxed{\underset{\sim}{\mathbb{V}} = \mathbb{N} \odot \mathbb{U}}$

$$\mathbb{M}=\begin{pmatrix} {}^4\underset{\sim}{\mathbf{A}}^{(11)} & {}^5\underset{\sim}{\mathbf{A}}^{(12)} & {}^6\underset{\sim}{\mathbf{A}}^{(13)} \\ ({}^5\underset{\sim}{\mathbf{A}}^{(12)})^T & {}^6\underset{\sim}{\mathbf{A}}^{(22)} & {}^7\underset{\sim}{\mathbf{A}}^{(23)} \\ ({}^6\underset{\sim}{\mathbf{A}}^{(13)})^T & ({}^7\underset{\sim}{\mathbf{A}}^{(23)})^T & {}^8\underset{\sim}{\mathbf{A}}^{(33)} \end{pmatrix},\ \mathbb{X}=\begin{pmatrix} \underset{\sim}{\boldsymbol{\varepsilon}} \\ \underset{\approx}{\boldsymbol{\gamma}} \\ \underset{\approx}{\boldsymbol{\varkappa}} \end{pmatrix},\ \mathbb{Y}=\begin{pmatrix} \underset{\sim}{\mathbf{P}} \\ \underset{\approx}{\mathbf{P}}^{(1)} \\ \underset{\approx}{\mathbf{P}}^{(2)} \end{pmatrix},$$

$$\mathbb{N}=\begin{pmatrix} {}^2\underset{\sim}{\mathbf{B}}^{(11)} & {}^3\underset{\sim}{\mathbf{B}}^{(12)} & {}^4\underset{\sim}{\mathbf{B}}^{(13)} \\ ({}^3\underset{\sim}{\mathbf{B}}^{(12)})^T & {}^4\underset{\sim}{\mathbf{B}}^{(22)} & {}^5\underset{\sim}{\mathbf{B}}^{(23)} \\ ({}^4\underset{\sim}{\mathbf{B}}^{(13)})^T & ({}^5\underset{\sim}{\mathbf{B}}^{(23)})^T & {}^6\underset{\sim}{\mathbf{B}}^{(33)} \end{pmatrix},\ \mathbb{U}=\begin{pmatrix} \mathbf{v} \\ \underset{\sim}{\mathbf{v}} \\ \underset{\approx}{\mathbf{v}} \end{pmatrix},\ \mathbb{V}=\begin{pmatrix} \boldsymbol{\pi} \\ \underset{\sim}{\boldsymbol{\pi}}^{(1)} \\ \underset{\approx}{\boldsymbol{\pi}}^{(2)} \end{pmatrix}$$

Here the following product-operators are introduced

$$\odot=\begin{pmatrix} \overset{2}{\otimes} & 0 & 0 \\ 0 & \overset{3}{\otimes} & 0 \\ 0 & 0 & \overset{4}{\otimes} \end{pmatrix},\quad \circledcirc=\begin{pmatrix} \overset{1}{\otimes} & 0 & 0 \\ 0 & \overset{2}{\otimes} & 0 \\ 0 & 0 & \overset{3}{\otimes} \end{pmatrix}$$

Let us present the elements of matrices as

$$
{}^4\underset{\sim}{\mathbf{A}}^{(11)} \overset{4}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\approx}{\mathbf{A}} \overset{4}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\boldsymbol{\varepsilon}}, \quad {}^6\underset{\sim}{\mathbf{A}}^{(22)} \overset{6}{\otimes} \underset{\approx}{\boldsymbol{\gamma}}\underset{\approx}{\boldsymbol{\gamma}} = \underset{\approx}{\mathbf{A}} \overset{4}{\otimes} (\underset{\approx}{\boldsymbol{\gamma}}^{\overset{1}{T}} \cdot \underset{\sim}{\mathbf{a}}^{(1)} \cdot \underset{\approx}{\boldsymbol{\gamma}}),
$$

$$
{}^8\underset{\sim}{\mathbf{A}}^{(33)} \overset{8}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}}\underset{\approx}{\boldsymbol{\varkappa}} = \underset{\approx}{\mathbf{A}} \overset{4}{\otimes} (\underset{\approx}{\boldsymbol{\varkappa}}^{T} \overset{2}{\otimes} \underset{\approx}{\mathbf{a}}^{(2)} \overset{2}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}}), \quad {}^5\underset{\sim}{\mathbf{A}}^{(12)} \overset{5}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\approx}{\boldsymbol{\gamma}} = \underset{\approx}{\bar{\mathbf{A}}} \overset{4}{\otimes} (\underset{\sim}{\boldsymbol{\varepsilon}}\mathbf{a} \cdot \underset{\approx}{\boldsymbol{\gamma}}),
$$

$$
{}^6\underset{\sim}{\mathbf{A}}^{(13)} \overset{6}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\approx}{\boldsymbol{\varkappa}} = \underset{\approx}{\bar{\mathbf{A}}} \overset{4}{\otimes} (\underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\mathbf{a}}^{(3)} \overset{2}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}}), \quad {}^7\underset{\sim}{\mathbf{A}}^{(23)} \overset{7}{\otimes} \underset{\approx}{\boldsymbol{\gamma}}\underset{\approx}{\boldsymbol{\varkappa}} = \underset{\approx}{\bar{\mathbf{A}}} \overset{4}{\otimes} (\underset{\approx}{\boldsymbol{\gamma}}^{\overset{1}{T}} \cdot \underset{\approx}{\mathbf{a}}^{(4)} \overset{2}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}});
$$

$$
{}^2\underset{\sim}{\mathbf{B}}^{(11)} \overset{2}{\otimes} \mathbf{v}\mathbf{v} = \underset{\sim}{\mathbf{B}} \overset{2}{\otimes} \mathbf{v}\mathbf{v}, \quad {}^4\underset{\sim}{\mathbf{B}}^{(22)} \overset{3}{\otimes} \underset{\sim}{\mathbf{v}}\underset{\sim}{\mathbf{v}} = \underset{\sim}{\mathbf{B}} \overset{2}{\otimes} (\underset{\sim}{\mathbf{v}}^{T} \cdot \underset{\sim}{\mathbf{b}}^{(1)} \cdot \underset{\sim}{\mathbf{v}}),
$$

$$
{}^6\underset{\sim}{\mathbf{B}}^{(33)} \overset{6}{\otimes} \underset{\approx}{\mathbf{v}}\underset{\approx}{\mathbf{v}} = \underset{\sim}{\mathbf{B}} \overset{2}{\otimes} (\underset{\approx}{\mathbf{v}}^{\overset{2}{T}} \overset{2}{\otimes} \underset{\approx}{\mathbf{b}}^{(2)} \overset{2}{\otimes} \underset{\approx}{\mathbf{v}}), \quad {}^3\underset{\sim}{\mathbf{B}}^{(12)} \overset{3}{\otimes} \mathbf{v}\underset{\sim}{\mathbf{v}} = \underset{\sim}{\bar{\mathbf{B}}} \overset{2}{\otimes} (\mathbf{v}\mathbf{b} \cdot \underset{\sim}{\mathbf{v}}),
$$

$$
{}^4\underset{\sim}{\mathbf{B}}^{(13)} \overset{4}{\otimes} \mathbf{v}\underset{\approx}{\mathbf{v}} = \underset{\sim}{\bar{\mathbf{B}}} \overset{2}{\otimes} (\mathbf{v}\underset{\sim}{\mathbf{b}}^{(3)} \overset{2}{\otimes} \underset{\approx}{\mathbf{v}}), \quad {}^5\underset{\sim}{\mathbf{B}}^{(23)} \overset{5}{\otimes} \underset{\sim}{\mathbf{v}}\underset{\approx}{\mathbf{v}} = \underset{\sim}{\bar{\mathbf{B}}} \overset{2}{\otimes} (\underset{\sim}{\mathbf{v}}^{T} \cdot \underset{\approx}{\mathbf{b}}^{(4)} \overset{2}{\otimes} \underset{\approx}{\mathbf{v}}),
$$

Here $\underset{\approx}{\mathbf{A}} = A_{ijkl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l$ is the elasticity modulus tensor, $A_{ijkl} = A_{klij} = A_{jikl}$
$\underset{\approx}{\bar{\mathbf{A}}} = \bar{A}_{ijkl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l,\ \bar{A}_{ijkl} = \bar{A}_{jikl} = \bar{A}_{ijlk},\ \bar{A}_{ijkl} \neq \bar{A}_{klij}$
$\mathbf{a} = a_i\mathbf{e}_i,\ \underset{\sim}{\mathbf{a}}^{(1)} = a_{ij}^{(1)}\mathbf{e}_i\mathbf{e}_j,\ \underset{\approx}{\mathbf{a}}^{(2)} = a_{ijkl}^{(2)}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l,\ \underset{\sim}{\mathbf{a}}^{(3)} = a_{ij}^{(3)}\mathbf{e}_i\mathbf{e}_j$ and
$\underset{\simeq}{\mathbf{a}}^{(4)} = a_{ijk}^{(4)}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$ are the length scale tensors

$a_{ij}^{(1)} = a_{ji}^{(1)},\ a_{ijkl}^{(2)} = a_{klij}^{(2)},\ a_{ij}^{(3)} =,\ a_{ijk}^{(4)} = a_{ikj}^{(4)}$
$\underset{\sim}{\mathbf{B}} = B_{ij}\mathbf{e}_i\mathbf{e}_j,\ \underset{\sim}{\bar{\mathbf{B}}} = \bar{B}_{ij}\mathbf{e}_i\mathbf{e}_j$
$\mathbf{b} = b_i\mathbf{e}_i,\ \underset{\sim}{\mathbf{b}}^{(1)} = b_{ij}^{(1)}\mathbf{e}_i\mathbf{e}_j,\ \underset{\approx}{\mathbf{b}}^{(2)} = b_{ijkl}^{(2)}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l,\ \underset{\sim}{\mathbf{b}}^{(3)} = b_{ij}^{(3)}\mathbf{e}_i\mathbf{e}_j$ and
$\underset{\simeq}{\mathbf{b}}^{(4)} = b_{ijk}^{(4)}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$ are the tensors for inertia effects

$b_{ij}^{(1)} \neq b_{ji}^{(1)},\ b_{ijkl}^{(2)} = b_{klij}^{(2)} = b_{jikl}^{(2)},\ B_{ij} = B_{ji},\ \bar{B}_{ij} \neq \bar{B}_{ji},\ b_{ij}^{(3)} = b_{ji}^{(3)},\ b_{ijk}^{(4)} = b_{ikj}^{(4)}$

$$\underset{\simeq}{\boldsymbol{\gamma}}^{\overset{1}{T}} = (\gamma_{ijk}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k)^{\overset{1}{T}} = \gamma_{ijk}\mathbf{e}_j\mathbf{e}_k\mathbf{e}_i$$

$$\underset{\simeq}{\mathbf{v}}^{\overset{2}{T}} = (v_{ijk}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k)^{\overset{2}{T}} = v_{ijk}\mathbf{e}_k\mathbf{e}_i\mathbf{e}_j$$

$\underset{\approx}{\boldsymbol{\varkappa}}^{T} = (\varkappa_{ijkl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l)^T = \varkappa_{ijkl}\mathbf{e}_k\mathbf{e}_l\mathbf{e}_i\mathbf{e}_j,\ \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$
δ_{ij} is the Kronecker delta, $i,j,k,l = 1,2,3$

$$\Phi(\underset{\sim}{\boldsymbol\varepsilon},\underset{\simeq}{\boldsymbol\gamma},\underset{\approx}{\boldsymbol\varkappa}) = \frac{1}{2}\Big[\underset{\approx}{\mathbf A}\overset{4}{\otimes}\Big(\underset{\sim}{\boldsymbol\varepsilon}\underset{\sim}{\boldsymbol\varepsilon} + \underset{\simeq}{\boldsymbol\gamma}^{\overset{1}{T}}\cdot\underset{\sim}{\mathbf a}^{(1)}\cdot\underset{\simeq}{\boldsymbol\gamma} + \underset{\approx}{\boldsymbol\varkappa}^{T}\overset{2}{\otimes}\underset{\approx}{\mathbf a}^{(2)}\overset{2}{\otimes}\underset{\approx}{\boldsymbol\varkappa}\Big)+$$

$$+2\underset{\approx}{\bar{\mathbf A}}\overset{4}{\otimes}\Big(\underset{\sim}{\boldsymbol\varepsilon}\mathbf a\cdot\underset{\simeq}{\boldsymbol\gamma} + \underset{\sim}{\boldsymbol\varepsilon}\underset{\sim}{\mathbf a}^{(3)}\overset{2}{\otimes}\underset{\approx}{\boldsymbol\varkappa} + \underset{\simeq}{\boldsymbol\gamma}^{\overset{1}{T}}\cdot\underset{\simeq}{\mathbf a}^{(4)}\overset{2}{\otimes}\underset{\approx}{\boldsymbol\varkappa}\Big)\Big] =$$

$$= \frac{1}{2}\Big[A^{ijkl}\Big(\varepsilon_{ij}\varepsilon_{kl} + \gamma_{mij}a^{(1)mn}\gamma_{nkl} + \varkappa_{mnij}a^{(2)mnpq}\varkappa_{pqkl}\Big)+$$

$$+2\bar{A}^{ijkl}\Big(\varepsilon_{ij}a^{m}\gamma_{mkl} + \varepsilon_{ij}a^{(3)mn}\varkappa_{mnkl} + \gamma_{mij}a^{(4)mnp}\varkappa_{npkl}\Big)\Big];$$

$$K(\mathbf v,\underset{\sim}{\mathbf v},\underset{\simeq}{\mathbf v}) = \frac{1}{2}\Big[\underset{\sim}{\mathbf B}\overset{2}{\otimes}\Big(\mathbf v\mathbf v + \underset{\sim}{\mathbf v}^{T}\cdot\underset{\sim}{\mathbf b}^{(1)}\cdot\underset{\sim}{\mathbf v} + \underset{\simeq}{\mathbf v}^{\overset{2}{T}}\overset{2}{\otimes}\underset{\approx}{\mathbf b}^{(2)}\overset{2}{\otimes}\underset{\simeq}{\mathbf v}\Big)+$$

$$+2\underset{\sim}{\bar{\mathbf B}}\overset{2}{\otimes}\Big(\mathbf v\mathbf b\cdot\underset{\sim}{\mathbf v} + \mathbf v\underset{\sim}{\mathbf b}^{(3)}\overset{2}{\otimes}\underset{\simeq}{\mathbf v} + \underset{\sim}{\mathbf v}^{T}\cdot\underset{\simeq}{\mathbf b}^{(4)}\overset{2}{\otimes}\underset{\simeq}{\mathbf v}\Big)\Big] =$$

$$= \frac{1}{2}\Big[B^{ij}\Big(v_i v_j + v_{ki}b^{(1)kl}v_{lj} + v_{kli}b^{(2)klmn}v_{mnj}\Big)+$$

$$+2\bar{B}^{ij}\Big(v_i b^{k}v_{kj} + v_i b^{(3)kl}v_{klj} + v_{ki}b^{(4)klm}v_{lmj}\Big)\Big];$$

$$\underset{\sim}{\mathbf{P}} = \frac{\partial\Phi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}} = \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}} + \underset{\approx}{\bar{\mathbf{A}}} \overset{2}{\otimes} (\mathbf{a}\cdot\underset{\sim}{\boldsymbol{\gamma}} + \underset{\sim}{\mathbf{a}}^{(3)} \overset{2}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}}),$$

$$\underset{\approx}{\mathbf{P}}^{(1)} = \frac{\partial\Phi}{\partial\nabla\underset{\sim}{\boldsymbol{\varepsilon}}} = \mathbf{a}\underset{\approx}{\bar{\mathbf{A}}}^{T} \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}} + \underset{\sim}{\mathbf{a}}^{(1)}\cdot\underset{\approx}{\boldsymbol{\gamma}} \overset{2}{\otimes} \underset{\approx}{\mathbf{A}} + (\underset{\approx}{\mathbf{a}}^{(4)} \overset{2}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}}) \overset{2}{\otimes} \underset{\approx}{\bar{\mathbf{A}}}^{T},$$

$$\underset{\approx}{\mathbf{P}}^{(2)} = \frac{\partial\Phi}{\partial\nabla\nabla\underset{\sim}{\boldsymbol{\varepsilon}}} = \underset{\sim}{\mathbf{a}}^{(3)}\underset{\approx}{\bar{\mathbf{A}}}^{T} \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}} + (\underset{\approx}{\mathbf{a}}^{(4)})^{\overset{1}{T}}\cdot\underset{\approx}{\boldsymbol{\gamma}} \overset{2}{\otimes} \underset{\approx}{\bar{\mathbf{A}}} + (\underset{\approx}{\mathbf{a}}^{(2)} \overset{2}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}}) \overset{2}{\otimes} \underset{\approx}{\mathbf{A}};$$

$$\boldsymbol{\pi} = \frac{\partial K}{\partial\mathbf{v}} = \underset{\sim}{\mathbf{B}}\cdot\mathbf{v} + \underset{\sim}{\bar{\mathbf{B}}}\cdot\left(\mathbf{b}\cdot\underset{\sim}{\mathbf{v}} + \underset{\sim}{\mathbf{b}}^{(3)} \overset{2}{\otimes} \underset{\approx}{\mathbf{v}}\right)$$

$$\underset{\sim}{\boldsymbol{\pi}}^{(1)} = \frac{\partial K}{\partial\nabla\mathbf{v}} = \mathbf{b}\underset{\sim}{\bar{\mathbf{B}}}^{T}\cdot\mathbf{v} + \frac{1}{2}\left(\underset{\sim}{\mathbf{b}}^{(1)} + (\underset{\sim}{\mathbf{b}}^{(1)})^{T}\right)\cdot\underset{\sim}{\mathbf{v}}\cdot\underset{\sim}{\mathbf{B}} + (\underset{\approx}{\mathbf{b}}^{(4)} \overset{2}{\otimes} \underset{\approx}{\mathbf{v}})\cdot\underset{\sim}{\bar{\mathbf{B}}}^{T},$$

$$\underset{\approx}{\boldsymbol{\pi}}^{(2)} = \frac{\partial K}{\partial\nabla\nabla\mathbf{v}} = \underset{\sim}{\mathbf{b}}^{(3)}\mathbf{v}\cdot\underset{\sim}{\bar{\mathbf{B}}} + \underset{\sim}{\bar{\mathbf{B}}}^{T}\cdot(\underset{\sim}{\mathbf{v}}^{T}\cdot\underset{\approx}{\mathbf{b}}^{(4)}) + (\underset{\approx}{\mathbf{b}}^{(2)} \overset{2}{\otimes} \underset{\approx}{\mathbf{v}})\cdot\underset{\sim}{\mathbf{B}}.$$

If the material has a center of symmetry, then the odd-rank material tensors are equal to zero

$$\Phi(\underset{\sim}{\boldsymbol{\varepsilon}},\underset{\approx}{\boldsymbol{\gamma}},\underset{\approx}{\boldsymbol{\varkappa}}) = \frac{1}{2}\underset{\approx}{\mathbf{A}} \overset{4}{\otimes} \left(\underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\boldsymbol{\varepsilon}} + \underset{\approx}{\boldsymbol{\gamma}}^{\overset{1}{T}} \cdot \underset{\sim}{\mathbf{a}}^{(1)} \cdot \underset{\approx}{\boldsymbol{\gamma}} + \underset{\approx}{\boldsymbol{\varkappa}}^{T} \overset{2}{\otimes} \underset{\approx}{\mathbf{a}}^{(2)} \overset{2}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}}\right) + \underset{\approx}{\bar{\mathbf{A}}} \overset{4}{\otimes} \left(\underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\mathbf{a}}^{(3)} \overset{2}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}}\right) =$$

$$= \frac{1}{2}A^{ijkl}\left(\varepsilon_{ij}\varepsilon_{kl} + \gamma_{mij}a^{(1)mn}\gamma_{nkl} + \varkappa_{mnij}a^{(2)mnpq}\varkappa_{pqkl}\right) + \bar{A}^{ijkl}\varepsilon_{ij}a^{(3)mn}\varkappa_{mnkl};$$

$$K(\mathbf{v},\underset{\sim}{\mathbf{v}},\underset{\approx}{\mathbf{v}}) = \frac{1}{2}\underset{\sim}{\mathbf{B}} \overset{2}{\otimes} \left(\mathbf{v}\mathbf{v} + \underset{\sim}{\mathbf{v}}^{T} \cdot \underset{\sim}{\mathbf{b}}^{(1)} \cdot \underset{\sim}{\mathbf{v}} + \underset{\approx}{\mathbf{v}}^{\overset{2}{T}} \overset{2}{\otimes} \underset{\approx}{\mathbf{b}}^{(2)} \overset{2}{\otimes} \underset{\approx}{\mathbf{v}}\right) + \underset{\sim}{\bar{\mathbf{B}}} \overset{2}{\otimes} \mathbf{v}\underset{\sim}{\mathbf{b}}^{(3)} \overset{2}{\otimes} \underset{\approx}{\mathbf{v}} =$$

$$= \frac{1}{2}B^{ij}\left(v_i v_j + v_{ki}b^{(1)kl}v_{lj} + v_{kli}b^{(2)klmn}v_{mnj}\right) + \bar{B}^{ij}v_i b^{(3)kl}v_{klj};$$

$$\underset{\sim}{\mathbf{P}} = \frac{\partial\Phi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}} = \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}} + \underset{\approx}{\bar{\mathbf{A}}} \overset{2}{\otimes} (\underset{\sim}{\mathbf{a}}^{(3)} \overset{2}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}}), \quad \underset{\approx}{\mathbf{P}}^{(1)} = \frac{\partial\Phi}{\partial\nabla\underset{\sim}{\boldsymbol{\varepsilon}}} = \underset{\sim}{\mathbf{a}}^{(1)} \cdot \underset{\approx}{\boldsymbol{\gamma}} \overset{2}{\otimes} \underset{\approx}{\mathbf{A}},$$

$$\underset{\approx}{\mathbf{P}}^{(2)} = \frac{\partial\Phi}{\partial\nabla\nabla\underset{\sim}{\boldsymbol{\varepsilon}}} = \underset{\sim}{\mathbf{a}}^{(3)}\underset{\approx}{\bar{\mathbf{A}}}^{T} \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}} + (\underset{\approx}{\mathbf{a}}^{(2)} \overset{2}{\otimes} \underset{\approx}{\boldsymbol{\varkappa}}) \overset{2}{\otimes} \underset{\approx}{\mathbf{A}}, \quad (\underset{\approx}{\mathbf{a}}^{(k)})^{T} = \underset{\approx}{\mathbf{a}}^{(k)};$$

$$\boldsymbol{\pi} = \frac{\partial K}{\partial\mathbf{v}} = \underset{\sim}{\mathbf{B}} \cdot \mathbf{v} + \underset{\sim}{\bar{\mathbf{B}}} \cdot (\underset{\sim}{\mathbf{b}}^{(3)} \overset{2}{\otimes} \underset{\approx}{\mathbf{v}}), \quad \underset{\sim}{\boldsymbol{\pi}}^{(1)} = \frac{\partial K}{\partial\nabla\mathbf{v}} = \frac{1}{2}\left(\underset{\sim}{\mathbf{b}}^{(1)} + (\underset{\sim}{\mathbf{b}}^{(1)})^{T}\right) \cdot \underset{\sim}{\mathbf{v}} \cdot \underset{\sim}{\mathbf{B}},$$

$$\underset{\approx}{\boldsymbol{\pi}}^{(2)} = \frac{\partial K}{\partial\nabla\nabla\mathbf{v}} = \underset{\sim}{\mathbf{b}}^{(3)}\mathbf{v} \cdot \underset{\sim}{\bar{\mathbf{B}}} + (\underset{\approx}{\mathbf{b}}^{(2)} \overset{2}{\otimes} \underset{\approx}{\mathbf{v}}) \cdot \underset{\sim}{\mathbf{B}}, \quad (\underset{\approx}{\mathbf{b}}^{(k)})^{T} = \underset{\approx}{\mathbf{b}}^{(k)}$$

If $\underset{\approx}{\bar{\mathbf{A}}} = 0$ and $\underset{\sim}{\bar{\mathbf{B}}} = 0$, then we get

$$\underset{\sim}{\mathbf{P}} = \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \underset{\sim}{\boldsymbol{\varepsilon}}, \quad \underset{\simeq}{\mathbf{P}}^{(1)} = \underset{\sim}{\mathbf{a}}^{(1)} \cdot \nabla \underset{\sim}{\mathbf{P}}, \quad \underset{\approx}{\mathbf{P}}^{(2)} = \underset{\approx}{\mathbf{a}}^{(2)} \overset{2}{\otimes} \nabla\nabla \underset{\sim}{\mathbf{P}};$$

$$\boldsymbol{\pi} = \underset{\sim}{\mathbf{B}} \cdot \mathbf{v}, \quad \underset{\sim}{\boldsymbol{\pi}}^{(1)} = \underset{\sim}{\mathbf{b}}^{(1)} \cdot \nabla \boldsymbol{\pi}, \quad \underset{\simeq}{\boldsymbol{\pi}}^{(2)} = \underset{\approx}{\mathbf{b}}^{(2)} \overset{2}{\otimes} \nabla\nabla \boldsymbol{\pi}.$$

Following Polizzotto (**C.Polizzotto** "A second strain gradient elasticity theory with second velocity gradient inertia"– Part I and II), we introduce the symmetric stress tensor $\underset{\sim}{\mathbf{P}}$ and the (basic) momentum vector (body momentum) $\mathbf{p}$ as a function of the velocity and velocity gradients, which in this case can be present as

$$\underset{\sim}{\mathbf{T}} = \underset{\sim}{\mathbf{P}} - \nabla \cdot \underset{\approx}{\mathbf{P}}^{(1)} + \nabla\nabla \overset{2}{\otimes} \underset{\approx}{\mathbf{P}}^{(2)} = (1 - \underset{\sim}{\mathbf{a}}^{(1)} \overset{2}{\otimes} \nabla\nabla + \underset{\approx}{\mathbf{a}}^{(2)} \overset{4}{\otimes} \nabla\nabla\nabla\nabla)\underset{\sim}{\mathbf{P}},$$

$$\mathbf{p} = \boldsymbol{\pi} - \nabla \cdot \underset{\sim}{\boldsymbol{\pi}}^{(1)} + \nabla\nabla \overset{2}{\otimes} \underset{\approx}{\boldsymbol{\pi}}^{(2)} = (1 - \underset{\sim}{\mathbf{b}}^{(1)} \overset{2}{\otimes} \nabla\nabla + \underset{\approx}{\mathbf{b}}^{(2)} \overset{4}{\otimes} \nabla\nabla\nabla\nabla)\boldsymbol{\pi}$$

Here $\underset{\sim}{\mathbf{P}}$ is the Cauchy stress tensor,
$\mathbf{p}$ is the momentum vector,
$\boldsymbol{\pi} = \rho\mathbf{v}$, $\underset{\sim}{\mathbf{B}} = \rho\underset{\sim}{\mathbf{E}}$, ρ is a density,
$\underset{\sim}{\mathbf{a}}^{(1)}$ and $\underset{\approx}{\mathbf{a}}^{(2)}$ are length scale tensors for statics,
$\underset{\sim}{\mathbf{b}}^{(1)}$ and $\underset{\approx}{\mathbf{b}}^{(2)}$ are length scale tensors for inertia effects (dynamics)

The case of an isotropic material

$\underset{\approx}{\mathbf{A}} = a_1 \underset{\approx}{\mathbf{C}}_{(1)} + a_2(\underset{\approx}{\mathbf{C}}_{(2)} + \underset{\approx}{\mathbf{C}}_{(3)})$, $\underset{\approx}{\bar{\mathbf{A}}} = 0$, $\underset{\sim}{\mathbf{a}}^{(1)} = l_1 \underset{\sim}{\mathbf{E}}$, $\underset{\approx}{\mathbf{a}}^{(2)} = l_2' \underset{\approx}{\mathbf{C}}_{(1)} + l_3'(\underset{\approx}{\mathbf{C}}_{(2)} + \underset{\approx}{\mathbf{C}}_{(3)})$,
$\underset{\sim}{\mathbf{B}} = \rho \underset{\sim}{\mathbf{E}}$, $\underset{\sim}{\bar{\mathbf{B}}} = 0$, $\underset{\sim}{\mathbf{b}}^{(1)} = d_1 \underset{\sim}{\mathbf{E}}$ and $\underset{\approx}{\mathbf{b}}^{(2)} = d_2' \underset{\approx}{\mathbf{C}}_{(1)} + d_3'(\underset{\approx}{\mathbf{C}}_{(2)} + \underset{\approx}{\mathbf{C}}_{(3)})$

Taking into account these relations we can represent the constitutive relations, the stress tensor $\underset{\sim}{\mathbf{T}}$ and the momentum vector $\mathbf{p}$ in the form

$$\underset{\sim}{\mathbf{P}} = \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \boldsymbol{\varepsilon}, \quad \underset{\simeq}{\mathbf{P}}^{(1)} = l_1 \nabla \underset{\sim}{\mathbf{P}}, \quad \underset{\approx}{\mathbf{P}}^{(2)} = (l_2' \underset{\sim}{\mathbf{E}} \Delta + 2 l_3' \nabla\nabla) \underset{\sim}{\mathbf{P}},$$

$$\boldsymbol{\pi} = \underset{\sim}{\mathbf{B}} \cdot \mathbf{v}, \quad \underset{\sim}{\boldsymbol{\pi}}^{(1)} = d_1 \nabla \boldsymbol{\pi}, \quad \underset{\simeq}{\boldsymbol{\pi}}^{(2)} = (d_2' \underset{\sim}{\mathbf{E}} \Delta + 2 d_3' \nabla\nabla) \boldsymbol{\pi},$$

$$\underset{\sim}{\mathbf{T}} = (1 - l_1 \Delta + l_2 \Delta^2) \underset{\sim}{\mathbf{P}}, \quad l_2 = l_2' + 2 l_3',$$
$$\mathbf{p} = (1 - d_1 \Delta + d_2 \Delta^2) \boldsymbol{\pi}, \quad d_2 = d_2' + d_3',$$

where l_1 and l_2 are length scale parameters for statics, d_1 and d_2 length scale parameters for inertia effects.

The case of a transversally-isotropic material

The linearly independent transversally-isotropic tensors of the second rank are the tensors

$$\underset{\sim}{\mathbf{I}} = \mathbf{e}_I \mathbf{e}_I, \quad \underset{\sim}{\boldsymbol{\epsilon}} = \epsilon_{IJ} \mathbf{e}_I \mathbf{e}_J, \quad \underset{\sim}{\boldsymbol{\gamma}}^{(3)} = \mathbf{e}_3 \mathbf{e}_3, \quad < I,J = 1,2 > .$$

Therefore, the general form of a transversal-isotropic tensor of the second rank $\underset{\sim}{\mathbf{a}}$, when components do not have any symmetry, is a linear combination of these tensors

$$\underset{\sim}{\mathbf{a}} = l_1 \underset{\sim}{\mathbf{I}} + l \underset{\sim}{\boldsymbol{\epsilon}} + l_2 \mathbf{e}_3 \mathbf{e}_3.$$

If $\underset{\sim}{\mathbf{a}}$ is a symmetric tensor ($\underset{\sim}{\mathbf{a}} = \underset{\sim}{\mathbf{a}}^T$), then $l = 0$ and from we get the following representation

$$\underset{\sim}{\mathbf{a}} = l_1 \underset{\sim}{\mathbf{I}} + l_2 \mathbf{e}_3 \mathbf{e}_3,$$

Any transversally-isotropic tensor of fourth rank $\underset{\approx}{\mathbf{A}}$ which components have the next symmetries $A_{ijkl} = A_{klij} = A_{ijlk}$, has five independent components and can be represented as

$$\begin{aligned}&\underset{\approx}{\mathbf{A}} = A_1\underset{\approx}{\mathbf{C}}_{(1)} + 2(A_1 - A_2)(\underset{\approx}{\mathbf{C}}_{(2)} + \underset{\approx}{\mathbf{C}}_{(3)}) + A_3(\underset{\sim}{\mathbf{I}}\mathbf{e}_3\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_3\underset{\sim}{\mathbf{I}}) +\\ &+ A_4\mathbf{e}_3\mathbf{e}_3\mathbf{e}_3\mathbf{e}_3 + A_5(\mathbf{e}_I\mathbf{e}_3\mathbf{e}_I\mathbf{e}_3 + \mathbf{e}_I\mathbf{e}_3\mathbf{e}_3\mathbf{e}_I + \mathbf{e}_3\mathbf{e}_J\mathbf{e}_J\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_J\mathbf{e}_3\mathbf{e}_J).\end{aligned}$$

Therefore, in this case, the stress tensor and the momentum vector will take the form

$$\begin{aligned}&\underset{\sim}{\mathbf{T}} = [1 + (l_3\Delta - l_1)\Delta + (l_4\Delta - l_2)\partial_3^2 + l_5\partial_3^4]\underset{\sim}{\mathbf{P}} = a\underset{\sim}{\mathbf{P}},\\ &a = 1 + (l_3\Delta - l_1)\Delta + (l_4\Delta - l_2)\partial_3^2 + l_5\partial_3^4\\ &\mathbf{p} = \rho[1 + (d_3\Delta - d_1)\Delta + (d_4\Delta - d_2)\partial_3^2 + d_5\partial_3^4]\mathbf{v} = \rho'\mathbf{v},\\ &\rho' = \rho[1 + (d_3\Delta - d_1)\Delta + (d_4\Delta - d_2)\partial_3^2 + d_5\partial_3^4]\\ &l_3 = l_1^{(2)} + 2l_2^{(2)}, \quad l_4 = 2(l_3^{(2)} + 2l_4^{(2)}), \quad l_5 = l_5^{(2)},\\ &d_3 = d_1^{(2)} + 2d_2^{(2)}, \quad d_4 = 2(d_3^{(2)} + 2d_4^{(2)}), \quad d_5 = d_5^{(2)}, \quad \Delta = \partial_I\partial_I.\end{aligned}$$

Equations of motion of second strain gradient elasticity theory with respect to displacement vector for the considered body can be written as

$$\Big(\nabla\cdot\underset{\sim}{\mathbf{T}}+\rho\mathbf{F}=\rho\frac{d\mathbf{p}}{dt}\Big)\Rightarrow$$
$$\Big((1-\underset{\sim}{\mathbf{a}}^{(1)}\overset{2}{\otimes}\nabla\nabla+\underset{\approx}{\mathbf{a}}^{(2)}\overset{4}{\otimes}\nabla\nabla\nabla\nabla)\nabla\cdot\underset{\sim}{\mathbf{P}}+\rho\mathbf{F}=(1-\underset{\sim}{\mathbf{b}}^{(1)}\overset{2}{\otimes}\nabla\nabla+\underset{\approx}{\mathbf{b}}^{(2)}\overset{4}{\otimes}\nabla\nabla\nabla\nabla)\frac{d\boldsymbol{\pi}}{dt}\Big).$$

This equations can be written in short form

$$\underset{\sim}{\mathbf{M}}'\cdot\mathbf{u}+\rho\mathbf{F}=0,$$

where we introduced

$$\begin{aligned}
&\underset{\approx}{\mathbf{A}}\overset{3}{\otimes}\nabla\nabla\mathbf{u}=\underset{\sim}{\mathbf{L}}\cdot\mathbf{u},\quad \underset{\sim}{\mathbf{L}}=\mathbf{e}_i\mathbf{e}_l A_{ijkl}\partial_j\partial_k,\\
&\underset{\sim}{\mathbf{M}}'=\underset{\sim}{\mathbf{L}}'-\rho'\underset{\sim}{\mathbf{E}}\partial_t^2,\quad \underset{\sim}{\mathbf{L}}'=a\underset{\sim}{\mathbf{L}},\\
&a=(1-\underset{\sim}{\mathbf{a}}^{(1)}\overset{2}{\otimes}\nabla\nabla+\underset{\approx}{\mathbf{a}}^{(2)}\overset{4}{\otimes}\nabla\nabla\nabla\nabla),\\
&\rho'=\rho(1-\underset{\sim}{\mathbf{b}}^{(1)}\overset{2}{\otimes}\nabla\nabla+\underset{\approx}{\mathbf{b}}^{(2)}\overset{4}{\otimes}\nabla\nabla\nabla\nabla).
\end{aligned}$$

Note that the last equation is the equation of motion with respect to the displacement vector of the second-gradient linear theory with respect to the strain tensor and the velocity vector for arbitrarily anisotropic homogeneous medium. $\underset{\sim}{\mathbf{M}}'$ is a second rank differential tensor operator.

$$\underset{\sim}{\mathbf{M}}' = \underset{\sim}{\mathbf{E}} Q_2' + (\lambda' + \mu')\nabla\nabla, \quad M_*' = (\underset{\sim}{\mathbf{E}} Q_1' - (\lambda' + \mu')\nabla\nabla) Q_2' = \underset{\sim}{\mathbf{N}}' Q_2', \; \mu' = \mu a,$$
$$\underset{\sim}{\mathbf{N}}' = \underset{\sim}{\mathbf{E}} Q_1' - (\lambda' + \mu')\nabla\nabla, \quad Q_1' = Q_2' + (\lambda' + \mu')\Delta = (\lambda' + 2\mu')\Delta - \rho'\partial_t^2, \; \lambda' = \lambda a,$$
$$Q_2' = \mu'\Delta - \rho'\partial_t^2, \quad a = 1 - l_1\Delta + l_2\Delta^2, \quad \rho' = \rho(1 - d_1\Delta + d_2\Delta^2), \; \underset{\sim}{\mathbf{N}}' \cdot \underset{\sim}{\mathbf{M}}' = \underset{\sim}{\mathbf{E}} Q_1' Q_2',$$

We obtain the following splitted equations

$$Q_1' Q_2' \mathbf{u} + \underset{\sim}{\mathbf{N}}' \cdot (\rho \mathbf{F}) = 0.$$

Equations of the quasi-static isotropic bodies with respect to displacement vector

$$a\Delta^2\mathbf{u} + \mathbf{G} = 0, \quad a = 1 - l_1\Delta + l_2\Delta^2,$$

$$\mathbf{G} = \frac{1}{(\lambda+\mu)\mu}\underset{\sim}{\mathbf{N}}\cdot(\rho\mathbf{F}), \quad \underset{\sim}{\mathbf{N}} = (\lambda+2\mu)\Delta - (\lambda+\mu)\nabla\nabla$$

which is reduced to the form

$$(l_2\Delta^4 - l_1\Delta^3 + \Delta^2)\mathbf{u} + \mathbf{G} = 0.$$

Equations of the quasi-static second strain gradient theory of isotropic prismatic bodies of constant thickness with respect to displacement vector and with respect to moments of the displacement vector

We consider a prismatic body of constant thickness $2h$ and take the middle plane as the base plane. Then the nabla-operator and the Laplacian we represent in the form

$$\hat{\nabla}\mathbb{F} = (\mathbf{r}^P\partial_P + \mathbf{r}^3\partial_3)\mathbb{F} = (\mathbf{r}^P\partial_P + h^{-1}\mathbf{n}\partial_3)\mathbb{F}, \quad -1 \le x^3 \le 1,$$
$$\Delta\mathbb{F} = (g^{PQ}\partial_P\partial_Q + g^{33}\partial_3^2)\mathbb{F} = (\bar{\Delta} + h^{-2}\partial_3^2)\mathbb{F}, \quad \bar{\Delta} = g^{PQ}\partial_P\partial_Q.$$

Thus, the equation for the prismatic bodies can be represented as

$$[(l_2\bar{\Delta}^2 - l_1\bar{\Delta} + 1)\bar{\Delta}^2 + h^{-2}(4l_2\bar{\Delta}^2 - 3l_1\bar{\Delta} + 2)\bar{\Delta}\partial_3^2 +$$
$$+h^{-4}(6l_2\bar{\Delta}^2 - 3l_1\bar{\Delta} - 1)\partial_3^4 + h^{-6}(4l_2\bar{\Delta} - l_1)\partial_3^6 + 2h^{-8}l_2\partial_3^8]\mathbf{u} + \mathbf{G} = 0.$$

Applying the kth moment operator of any system of orthogonal polynomials (Legendre, Chebyshev) to the equation, we obtain the following equations in moments of displacement vector

$$(l_2\bar{\Delta}^2 - l_1\bar{\Delta} + 1)\bar{\Delta}^2\overset{(k)}{\mathbf{u}} + h^{-2}(4l_2\bar{\Delta}^2 - 3l_1\bar{\Delta} + 2)\bar{\Delta}\overset{(k)}{\mathbf{u}}{}'' + h^{-4}(6l_2\bar{\Delta}^2 - 3l_1\bar{\Delta} - 1)\overset{(k)}{\mathbf{u}}{}^{IV} +$$
$$+h^{-6}(4l_2\bar{\Delta} - l_1)\overset{(k)}{\mathbf{u}}{}^{VI} + 2h^{-8}l_2\partial_3^8\overset{(k)}{\mathbf{u}}{}^{VIII} + \mathbf{G} = 0, \quad k\in\mathbb{N}_0 \ (\mathbb{N}_0=\{0,1,2,\ldots\}).$$

Equations of the quasi-static second strain gradient theory of isotropic prismatic bodies of constant thickness in moments of displacement vector with respect to the system of Legendre polynomials

To obtain the desired systems of equations we need to find expression for $\overset{(k)}{\mathbf{u}}{}''$, $\overset{(k)}{\mathbf{u}}{}^{IV}$, $\overset{(k)}{\mathbf{u}}{}^{VI}$ and $\overset{(k)}{\mathbf{u}}{}^{VIII}$ when $-1 \leq x^3 \leq 1$.
In this case they are defined using the following relationship

$$\overset{(n)}{\mathbf{u}}{}^{(2m)}(x') = (2n+1)\sum_{k=1}^{\infty} C^{2m-1}_{k+2m-2}\prod_{s=1}^{2m-1}(2n+2k+2s-1)\overset{(n+2k+2m-2)}{\mathbf{u}}$$

$$= \frac{2n+1}{2}\sum_{k=1}^{2m}(-1)^{k+1}[(\partial_3^{2m-k}\mathbf{u})^+ + (-1)^{n+k}(\partial_3^{2m-k}\mathbf{u})^-]P_n^{k-1}(1) + \underline{\overset{(k)}{\mathbf{u}}}{}^{(2m)},$$

$$\underline{\overset{(k)}{\mathbf{u}}}{}^{(2m)} = (2n+1)\sum_{k=1}^{[n/2-m+1]} C^{2m-1}_{k+2m-2}\prod_{s=1}^{2m-1}(2n-2k-2s+3)\overset{(n-2k-2m+2)}{\mathbf{u}}.$$

Here $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $\mathbb{N}$ is the set of natural numbers, C^{2m-1}_{k+2m-2} are binomial coefficients, $(\partial_3^s\mathbf{u})^{\pm} = (\partial_3^s\mathbf{u})\big|_{x^3=\pm 1}$, $s\in\mathbb{N}$, $n\in\mathbb{N}_0$, $m\in\mathbb{N}$.

We see that $\overset{(k)}{\mathbf{u}}{}''$, $\overset{(k)}{\mathbf{u}}{}^{IV}$, …, are represented as an infinite sum of moments of displacement vector. So, we get different representations of the systems of equations of the quasi-static second strain gradient theory of isotropic prismatic thin bodies of constant thickness in moments.

Equations of the quasi-static second strain gradient elasticity theory of transversal-isotropic bodies with respect to displacement vector

In case of transversal-isotropic medium the tensor-operator $\underset{\sim}{\mathbf{L}}$, tensor-operator o algebraic cofactors $\underset{\sim}{\mathbf{L}}_*$ for $\underset{\sim}{\mathbf{L}}$ and the determinant $|\underset{\sim}{\mathbf{L}}|$ have the form

$$
\begin{aligned}
&2\underset{\sim}{\mathbf{L}} = \big[(A_1 - A_2)\underset{\sim}{\mathbf{I}} + 2A_5\mathbf{e}_3\mathbf{e}_3\big]\bar{\Delta} + (A_1 + A_2)\nabla^0\nabla^0 + \\
&\quad +2(A_3 + A_5)[\mathbf{e}_3\nabla^0 + (\mathbf{e}_3\nabla^0)^T]\partial_3 + 2(A_5\underset{\sim}{\mathbf{I}} + A_4\mathbf{e}_3\mathbf{e}_3)\partial_3^2; \\
&2\underset{\sim}{\mathbf{L}}_* = 2\underset{\sim}{\mathbf{I}}\{A_1A_5\bar{\Delta}^2 + [A_1A_4 - A_3(A_3 + 2A_5)]\bar{\Delta}\partial_3^2 + A_4A_5\partial_3^4\} - \\
&\quad -\{(A_1 + A_2)A_5\bar{\Delta} + [(A_1 + A_2)A_4 - 2(A_3 + A_5)^2]\partial_3^2\}\nabla^0\nabla^0 - \\
&\quad -[(A_1 - A_2)(A_3 + A_5)\bar{\Delta} + 2(A_3 + A_5)A_5\partial_3^2][\mathbf{e}_3\nabla^0 + (\mathbf{e}_3\nabla^0)^T]\partial_3 + \\
&\quad +\underset{\sim}{\mathbf{e}}_3[A_1(A_1 - A_2)\bar{\Delta}^2 + (3A_1 - A_2)A_5\bar{\Delta}\partial_3^2 + 2A_5^2\partial_3^4]; \\
&\underset{\sim}{\mathbf{L}} \cdot \underset{\sim}{\mathbf{L}}_* = \underset{\sim}{\mathbf{L}}_* \cdot \underset{\sim}{\mathbf{L}} = \underset{\sim}{\mathbf{E}}|\underset{\sim}{\mathbf{L}}|,\ |\underset{\sim}{\mathbf{L}}| = \det\underset{\sim}{\mathbf{L}},\ \nabla^0 = \mathbf{e}_I\partial_I,\ \underset{\sim}{\mathbf{I}} = \mathbf{e}_I\mathbf{e}_I,\ \bar{\Delta} = \partial_I\partial_I; \\
&|\underset{\sim}{\mathbf{L}}| = A\bar{\Delta}^3 + B\bar{\Delta}^2\partial_3^2 + C\bar{\Delta}\partial_3^4 + D\partial_3^6 = k(\bar{\Delta} + k_1\partial_3^2)(\bar{\Delta} + k_2\partial_3^2)(\bar{\Delta} + k_3\partial_3^2); \\
&k = A,\ k_1 + k_2 + k_3 = B/A,\ k_1k_2 + k_1k_3 + k_2k_3 = C/A,\ k_1k_2k_3 = D/A, \\
&A = (1/2)(A_1 - A_2)A_1A_5,\ B = (1/2)\{(A_1 - A_2)[A_1A_4 - A_3(A_3 + 2A_5)] + 2A_1A_5^2\}, \\
&C = (1/2)[(3A_1 - A_2)A_4A_5 - 2A_3A_5(A_3 + 2A_5)],\ D = A_4A_5^2.
\end{aligned}
$$

Neglecting inertial terms we will have

$$\boxed{\underset{\sim}{\mathbf{L}}' \cdot \mathbf{u} + \rho \mathbf{F} = 0, \quad \underset{\sim}{\mathbf{L}}' = a \underset{\sim}{\mathbf{L}}.}$$

The splitted equations of the classic theory of elasticity are

$$\boxed{|\underset{\sim}{\mathbf{L}}|\mathbf{u} + \underset{\sim}{\mathbf{L}}_* \cdot (\rho \mathbf{F}) = 0}$$

or if we will look for a solution as $\mathbf{u} = \underset{\sim}{\mathbf{L}}_* \cdot \mathbf{v}$ where $\mathbf{v}$ is an arbitrary vector, then we get

$$\boxed{|\underset{\sim}{\mathbf{L}}|\mathbf{v} + \rho \mathbf{F} = 0.}$$

Multiplying the equation by $\underset{\sim}{\mathbf{L}}_*$ on the left, we get

$$\boxed{|\underset{\sim}{\mathbf{L}}|'\mathbf{u} + \underset{\sim}{\mathbf{L}}_* \cdot (\rho \mathbf{F}) = 0, \quad |\underset{\sim}{\mathbf{L}}|' = a|\underset{\sim}{\mathbf{L}}|, \quad a = 1 + (l_3\bar{\Delta} - l_1)\bar{\Delta} + (l_4\bar{\Delta} - l_2)\partial_3^2 + l_5\partial_3^4.}$$

Equations of the quasi-static second strain gradient elasticity theory of transversal-isotropic bodies with respect to the displacement vector

Taking into account the relation for $|\underset{\sim}{\mathbf{L}}|$ we obtain the following vector equation splitted over the components of the displacement vector

$$\begin{aligned}&\{A(l_3\bar{\Delta}^2 - l_1\bar{\Delta} + 1)\bar{\Delta}^3 + [(Bl_3 + Al_4)\bar{\Delta}^2 - (Bl_1 + Al_2)\bar{\Delta} + B]\bar{\Delta}^2\partial_3^2 + \\ &\quad + [(Al_5 + Bl_4 + Cl_3)\bar{\Delta}^2 - (Bl_2 + Cl_1)\bar{\Delta} + C]\bar{\Delta}\partial_3^4 + [(Bl_5 + Cl_4 + Dl_3)\bar{\Delta}^2 - \\ &\quad - \big(Cl_2 + D(l_1 - l_4)\big)\bar{\Delta} + D(1 - l_2)]\partial_3^6 + Cl_5\bar{\Delta}\partial_3^8 + Dl_5\partial_3^{10}\}\mathbf{u} + \underset{\sim}{\mathbf{L}}_*\cdot(\rho\mathbf{F}) = 0,\end{aligned}$$

Equations of the quasi-static second strain gradient theory of transversal-isotropic prismatic bodies of constant thickness with respect to displacement vector and with respect to moments of the displacement vector

Here we have the next equation for prismatic bodies

$$\{A(l_3\bar{\Delta}^2 - l_1\bar{\Delta} + 1)\bar{\Delta}^3 + h^{-2}[(Bl_3 + Al_4)\bar{\Delta}^2 - (Bl_1 + Al_2)\bar{\Delta} + B]\bar{\Delta}^2\partial_3^2 + \\ + h^{-4}[(Al_5 + Bl_4 + Cl_3)\bar{\Delta}^2 - (Bl_2 + Cl_1)\bar{\Delta} + C]\bar{\Delta}\partial_3^4 + h^{-6}[(Bl_5 + Cl_4 + Dl_3)\bar{\Delta}^2 - \\ -\big(Cl_2 + D(l_1 - l_4)\big)\bar{\Delta} + D(1 - l_2)]\partial_3^6 + h^{-8}Cl_5\bar{\Delta}\partial_3^8 + h^{-10}Dl_5\partial_3^{10}\}\mathbf{u} + \underset{\sim}{\mathbf{L}}_*\cdot(\rho\mathbf{F}) = 0.$$

Applying the kth moment operator of any system of orthogonal polynomials (Legendre, Chebyshev) to the equation, we obtain the following equations in moments of displacement vector

$$A(l_3\bar{\Delta}^2 - l_1\bar{\Delta} + 1)\bar{\Delta}^3\overset{(k)}{\mathbf{u}} + h^{-2}[(Bl_3 + Al_4)\bar{\Delta}^2 - (Bl_1 + Al_2)\bar{\Delta} + B]\bar{\Delta}^2\overset{(k)}{\mathbf{u}}'' + \\ + h^{-4}[(Al_5 + Bl_4 + Cl_3)\bar{\Delta}^2 - (Bl_2 + Cl_1)\bar{\Delta} + C]\bar{\Delta}\overset{(k)}{\mathbf{u}}^{IV} + h^{-6}[(Bl_5 + Cl_4 + Dl_3)\bar{\Delta}^2 - \\ -\big(Cl_2 + D(l_1 - l_4)\big)\bar{\Delta} + D(1 - l_2)]\overset{(k)}{\mathbf{u}}^{VI} + h^{-8}Cl_5\bar{\Delta}\overset{(k)}{\mathbf{u}}^{VIII} + h^{-10}Dl_5\overset{(k)}{\mathbf{u}}^{X} + \\ + \underset{\sim}{\mathbf{L}}_*\cdot(\rho\mathbf{F}) = 0, \quad k\in\mathbb{N}_0\ (\mathbb{N}_0=\{0,1,2,\ldots\}).$$

Equations of the quasi-static second strain gradient theory of transversal-isotropic prismatic bodies of constant thickness in moments of displacement vector relative to the system of Legendre polynomials

To obtain the desired systems of equations we need to find expression for $\overset{(k)}{\mathbf{u}}{}''$, $\overset{(k)}{\mathbf{u}}{}^{IV}$, $\overset{(k)}{\mathbf{u}}{}^{VI}$, $\overset{(k)}{\mathbf{u}}{}^{VIII}$ and $\overset{(k)}{\mathbf{u}}{}^{X}$ when $-1 \leq x^3 \leq 1$. Here they are defined using the following relationship

$$\overset{(n)}{\mathbf{u}}{}^{(2m)}(x')=(2n+1)\sum_{k=1}^{\infty} C_{k+2m-2}^{2m-1}\prod_{s=1}^{2m-1}(2n+2k+2s-1)\overset{(n+2k+2m-2)}{\mathbf{u}}$$

$$=\frac{2n+1}{2}\sum_{k=1}^{2m}(-1)^{k+1}[(\partial_3^{2m-k}\mathbf{u})^{+}+(-1)^{n+k}(\partial_3^{2m-k}\mathbf{u})^{-}]P_n^{k-1}(1)+\overset{(k)}{\underline{\mathbf{u}}}{}^{(2m)},$$

$$\overset{(k)}{\underline{\mathbf{u}}}{}^{(2m)} = (2n+1)\sum_{k=1}^{[n/2-m+1]} C_{k+2m-2}^{2m-1}\prod_{s=1}^{2m-1}(2n-2k-2s+3)\overset{(n-2k-2m+2)}{\mathbf{u}}, \; n\in\mathbb{N}_0,\, m\in\mathbb{N}.$$

Here $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $\mathbb{N}$ is the set of natural numbers, C_{k+2m-2}^{2m-1} are binomial coefficients, $(\partial_3^s\mathbf{u})^- = (\partial_3^s\mathbf{u})\big|_{x^3=-1}$ and $(\partial_3^s\mathbf{u})^+ = (\partial_3^s\mathbf{u})\big|_{x^3=1}$, $s\in\mathbb{N}$.

Eigenvalue Problem and Construction of a Complete System of Eigentensor Columns of Symmetric Tensor-Block Matrix

Above, we have written the relations using tensor-block matrices (TBM). Let me introduce a definition of the TBM.

Block matrix, whose blocks are composed of the various rank tensors, is called the TBM.

TBM of sizes $q \times m$ can be written as

$$\mathbb{M} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{A}_{q1} & \mathbf{A}_{q2} & \cdots & \mathbf{A}_{qm} \end{pmatrix}, \tag{8.1}$$

where m and q are some natural numbers; $\mathbf{A}_{kl}$, $k = \overline{1,q}$, $l = \overline{1,m}$ are the arbitrary tensors, also called the subtensors of TBM (8.1)

The matrix

$$\mathbb{M}^T = \begin{pmatrix} \mathbf{A}^T_{11} & \mathbf{A}^T_{21} & \cdots & \mathbf{A}^T_{q1} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{A}^T_{1m} & \mathbf{A}_{2m} & \cdots & \mathbf{A}^T_{qm} \end{pmatrix},$$

is called transposed matrix with the TBM.

TBM, which coincides with its transpose matrix is called symmetric.

TBM is said to be quadratic if the number of rows and columns are the same ($q = m$).
Such matrices are often used in the application.
Column matrix (row matrix) whose elements are the tensors of various rank, is called the tensor column (tensor row). The tensor column $\underset{\sim}{\mathbb{U}}$ whose elements are the p-rank tensors, is represented in the form

$$\mathbb{U} = (\mathbf{U}_1, \ldots, \mathbf{U}_m)^T = \left(\mathrm{U}_{1,\, i_1 i_2 \cdots i_p}, \ldots, \mathrm{U}_{m,\, i_1 i_2 \cdots i_p}\right)^T \mathbf{R}^{i_1 i_2 \cdots i_p}.$$

Now we can formulate the eigenvalue problem of the TBM

Eigenvalue problem. It is required to find all tensor columns

$$\mathbb{U} = (\mathbf{U}_1, \ldots, \mathbf{U}_m)^T \quad \left(\mathbb{U}^T = (\mathbf{U}_1, \ldots, \mathbf{U}_m)\right),$$

satisfying the equation

$$\mathbb{M} \overset{p}{\otimes} \mathbb{U} = \lambda \mathbb{U},$$

where λ is a scalar.

1. The authors propose constitutive relations and derive the equations of motion and equilibrium of the second-gradient linear theory of anisotropic inhomogeneous elastic bodies with respect to the strain tensor and the three-dimensional velocity vector. These relations generalize the corresponding relations obtained by C.Polizzotto.
2. Authors derived the equations of motion and equilibrium of the second-gradient linear theory of anisotropic homogeneous elastic bodies with respect to the displacement vector. At the same time, the authors presented these equations using the introduced differential tensor-operators of the second rank. As a special case, equations are obtained for the cases of isotropic and transversely-isotropic bodies.
3. For the differential tensor-operators indicated above, we have found expressions for the differential tensor-operators of the cofactors. Based on them, the corresponding equations splitted over the components of the displacement vector are obtained. 4. In the case of isotropic and transversely-isotropic homogeneous elastic bodies with respect to the displacement vector, three-dimensional splitted equations for prismatic thin bodies with respect to the displacement vector are obtained. To do this, the authors used the classical parametrization of the domain of a prismatic thin body.

5. Applying the method of orthogonal polynomials from the splitted quasi-static equations indicated above, the quasi-static equations of the second-gradient theory of isotropic and transversely isotropic homogeneous elastic prismatic thin bodies in moments of displacement vector with respect to an arbitrary orthogonal system of polynomials are obtained. As a special case, we obtain equations in moments with respect to the system of Legendre polynomials.

It should be noted that the resulting systems of equations of various approximations in moments of the displacement vector are splitted for each moment of the displacement vector. As a result, we come to the high-order elliptic-type equations. The Vekua method can be used to solve these equations. Thus, one can obtain analytical solutions in the class of analytic functions of a complex variable. A special case of the general solution of the 6-th order elliptic equation can be found in the work
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