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Abstract

The paper presents an algorithm for constructing a rapidly converging series representing a generalized solution of a mixed problem for a telegraphic equation considered in a half-band. Reviewed the case of an essentially non-self-adjoint operator in a spatial variable. The system of root functions of a differential operator, in addition to its eigenfunctions, contains an infinite number of associated functions. The constructed series can be considered as a generalized d'Alembert formula. A new axiomatic A.P. Khromov's method is applied to construct the solution. The proposed approach superseds the traditional method of separating variables for solving mixed problems, which usually results in to slowly converging series. For the problem under consideration, in general, the method of separating variables is not applicable, since the coefficient of the equation depends both on the spatial variable and on time.

Keywords (separated by '-'): Telegraph equation - Mixed problem - Generalized d'Alembert formula - Fourier method - Non-self-adjoint operator - Divergent series

Axiomatic Method for Constructing a Generalized Solution of a Mixed Problem for a Telegraph Equation

Igor S. Lomov

Abstract The paper presents an algorithm for constructing a rapidly converging series representing a generalized solution of a mixed problem for a telegraphic equation considered in a half-band. Reviewed the case of an essentially non-self-adjoint operator in a spatial variable. The system of root functions of a differential operator, in addition to its eigenfunctions, contains an infinite number of associated functions. The constructed series can be considered as a generalized d'Alembert formula. A new axiomatic A.P. Khromov's method is applied to construct the solution. The proposed approach superseds the traditional method of separating variables for solving mixed problems, which usually results in to slowly converging series. For the problem under consideration, in general, the method of separating variables is not applicable, since the coefficient of the equation depends both on the spatial variable and on time.

Keywords Telegraph equation · Mixed problem · Generalized d'Alembert formula · Fourier method · Non-self-adjoint operator · Divergent series

1 Introduction

A number of mathematical models used in problems of sound theory (elasticity), light, electricity and magnetism, contain the so-called telegraph equation $u_{tt}(x,t) = u_{xx}(x,t) - qu(x,t)$. A mixed problem is posed. Consider the case when the potential q can also depend on time, $q = q(x,t)$. To construct a solution to a generalized mixed problem, we use the recently developed axiomatic method of A.P. Khromov [1]. Previously, he developed a sequential method for constructing a generalized solution to the problem under consideration [2, 3]. The advantage of these methods over the methods used earlier consist in the fact that minimum requirements are imposed on the initial data of the problem, the justification of the result attracts a

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minimum number of additional statements, and the solution is given in the form of a rapidly converging functional series.

Let's consider four problems sequentially, for which we will find generalized solutions.

2 A Mixed Problem for a Homogeneous Wave Equation with a Nonzero Initial Deviation

Consider the following problem

$$u_{tt}(x,t) = u_{xx}(x,t), \quad (x,t) \in (0,1) \times (0,+\infty), \tag{1}$$

$$u(0,t) = 0, \quad {}_x(0,t) = u_x(1,t), \quad t \geq 0, \tag{2}$$

$$u(x,0) = \varphi(x), \;\; u_t(x,0) = 0, \quad x \in [0,1], \tag{3}$$

$\varphi(x)$—complex-valued, integrable on $(0,1)$ functions, $\varphi(x) \in \mathcal{L}(0,1)$. We use the notation derivatives $u_x = \frac{\partial u}{\partial x}$, etc.

The peculiarity of the problem (1)–(3) is due to the fact that the corresponding Sturm–Liouville operator $L_0 : \quad ly = -y''(x),\; x \in (0,1),\, y(0) = 0,\, y'(0) = y'(1)$, is essentially non-self-adjoint (according to Ilyin)—the system of root functions of this operator, in addition to its eigenfunctions, contains an infinite number of associated functions (the Samarsky-Ionkin problem). Let's write out this system.

Denote by ϱ_k the square roots of the eigenvalues operator, $\{u_k(x)\}$—system of eigen and associated operator functions, moreover, $u_{2k-1}(x)$—eigenfunctions, $u_{2k}(x)$—associated functions, $k \geq 1$, $\{v_k(x)\}$—biorthogonally conjugate system of functions, $(u_k, v_n) = \delta_{kn} = \begin{cases} 1, k = n, \\ 0, k \neq n \end{cases}$, where $(u_k, v_n) = \int_0^1 u_k(x) v_n(x) dx$.

Then $\varrho_0 = 0,\; \varrho_{2k-1} = \varrho_{2k} = 2\pi k,\; k \geq 1, \quad u_0(x) = x,\; v_0(x) = 2,\; u_{2k-1}(x) = \sin 2\pi k x,\; v_{2k-1}(x) = 4(1-x)\sin 2\pi k x, \quad u_{2k}(x) = -\frac{x}{4\pi k}\cos 2\pi k x,\; v_{2k}(x) = -16\pi k \cos 2\pi k x$. So the chosen system $\{u_k(x)\}$ of root functions of the operator forms unconditional basis in the space $\mathcal{L}^2(0,1)$. System $\{v_k(x)\}$ also forms an unconditional basis in this space.

The formal solution of the problem (1)–(3) by the Fourier method is

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}\Big\{2(x+t)(1,\varphi) + \\ &+4\sum_{n=1}^{\infty}\big[(\varphi(\tau),(1-\tau)\sin 2\pi n\tau)\sin 2\pi n(x+t) + \\ &+(\varphi(\tau),\cos 2\pi n\tau)(x+t)\cos 2\pi n(x+t)\big] + \\ &+2(x-t)(1,\varphi) + 4\sum_{n=1}^{\infty}\big[(\varphi(\tau),(1-\tau)\sin 2\pi n\tau)\sin 2\pi n(x-t) + \\ &+(\varphi(\tau),\cos 2\pi n\tau)(x-t)\cos 2\pi n(x-t)\big]\Big\}. \end{aligned} \tag{4}$$

Definition 1 By the classical solution (almost everywhere solution) of the problem (1)–(3) we mean the function $u(x, t)$ continuous and continuously differentiable with respect to x and t in half-strip $[0, 1] \times [0, \infty)$, and the functions $u_x(x, t)$, $u_t(x, t)$ are absolutely continuous in $x \in [0, 1]$ and $t \in [0, \infty)$, respectively, satisfying the conditions (2), (3) and almost everywhere in x and t the Eq. (1).

Let us present a uniqueness theorem for the classical solution of the problem (1)–(3). Fix an arbitrary number $T > 0$, let Q_T—rectangle, $Q_T = [0, 1] \times [0, T]$, denoted by Q is the class of functions integrable on Q_T, $f \in Q \Leftrightarrow f(x, t) \in \mathcal{L}(Q_T)$.

Theorem 1 *If $u(x, t)$ is a classical solution to the problem (1)–(3) with condition $u_{tt}(x, t) \in Q$ ($\forall T > 0$), then it is unique and can be found by the formula (4), in which the series on the right for any fixed $t > 0$ converge absolutely and uniformly in $x \in [0, 1]$.*

The proof of the theorem follows the scheme described in [4] and does not depends on specific boundary conditions.

Note that the series (4) makes sense for any function $\varphi(x) \in \mathcal{L}(0, 1)$, although now it can also be divergent. Nevertheless, we will assume that it is a *formal solution* of the problem (1)–(3), but now understood *purely formally*. This problem (1)–(3) will be called the *generalized mixed problem*. Finding a solution to a generalized mixed problem means finding the "sum" of, generally speaking, a divergent series. "Sum" in quotes means that this is the sum of a divergent (generally) series (see [5, p. 101], [6, p. 6, 19]).

Finding a solution to the generalized mixed problem (1)–(3) means finding the "sum" of the divergent series (4).

In addition to the three axioms about divergent series [6, p. 19], following A.P. Khromov, we will also use the following integration rule for a divergent series:

$$\int \sum = \sum \int, \tag{5}$$

where $\int$ is a definite integral.

Let's go back to the row (4). Before transforming it, let us write the formal expansion of the function $\varphi(x)$ into a series in terms of the root system functions of the operator L_0:

$$\varphi(x) \sim 2x(1, \varphi) + 4\sum_{n=1}^{\infty}[(\varphi(\tau), (1-\tau)\sin 2\pi n\tau)\sin 2\pi nx + \\ +(\varphi(\tau), \cos 2\pi n\tau)x\cos 2\pi nx]. \tag{6}$$

The series (4) can be represented as

$$u(x, t) = \sum\nolimits_{+} + \sum\nolimits_{-}, \tag{7}$$

where $\sum_{\pm} = \sum\limits_{n=1}^{\infty} \ldots (x \pm t)$. Comparing (6), (7), we conclude that to find the "sum" of the series (4), we need to find the "sum" of the series (6) .

Let the "sum" of the series (6) for $x \in [0, 1]$ be some function $g(x) \in \mathcal{L}(0, 1)$. Then, in accordance with rule (5), we have

$$\begin{aligned} &\int\limits_0^x g(\eta)d\eta = 2(1, \varphi)\int\limits_0^x \eta d\eta + \\ &+4\sum_{n=1}^{\infty}\Big[(\varphi(\tau), (1-\tau)\sin 2\pi n\tau)\int\limits_0^x \sin 2\pi n\eta d\eta + \\ &+(\varphi(\tau), \cos 2\pi n\tau)\int\limits_0^x \eta\cos 2\pi n\eta d\eta\Big], \quad x \in [0, 1]. \end{aligned} \tag{8}$$

The following generalization to the considered system $\{u_k(x)\}$ of Lebesgue's theorem on term-by-term integration of the trigonometric Fourier series takes place.

Theorem 2 *Let a function $\varphi(x) \in \mathcal{L}(0, 1)$ be given that has the series (6) as its biorthogonal expansion in the system $\{u_k(x)\}$. If the segment is $[A, B] \subseteq [0, 1]$, then*

$$\begin{aligned} &\int\limits_A^B \varphi(x)dx = \int\limits_A^B 2x(1, \varphi)dx + \sum_{n=1}^{\infty}\int\limits_A^B [4(\varphi(\tau), (1-\tau)\sin 2\pi n\tau)\sin 2\pi nx + \\ &+4(\varphi(\tau), \cos 2\pi n\tau)x\cos 2\pi nx]dx. \end{aligned}$$

Those, the biorthogonal series (6) can be integrated term-by-term, the resulting series converges and its sum is equal to $\int\limits_A^B \varphi(x)dx$. In this case, the series (6) itself may not converge.

The proof of Theorem 2 is carried out in Sect. 5.

According to Theorem 2, the sum of the series (8), the usual sum, is the function $\int\limits_0^x \varphi(\eta)d\eta$. But then, $\int\limits_0^x g(\eta)d\eta = \int\limits_0^x \varphi(\eta)d\eta$, i.e. $g(x) = \varphi(x)$ is true almost everywhere on the interval $[0, 1]$, we have found the "sum" of the series (6), which can also be divergent.

The formal series (6) is defined for all values of $x \in \mathbf{R}$. Denote by $\widetilde{\varphi}(x)$ the "sum" of the series (6) for all values of $x \in \mathbf{R}$. By virtue of (6) and (7) we conclude that the "sum" $u(x, t)$ of the series (4) is a function

$$u(x, t) = \frac{1}{2}[\widetilde{\varphi}(x+t) + \widetilde{\varphi}(x-t)]. \tag{9}$$

Proven

Theorem 3 *The solution of the generalized mixed problem (1)–(3) is the function $u(x, t)$ from the class Q defined by the formula (9).*

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Let us find an algorithm for extending the function $\widetilde{\varphi}(x)$ from the segment [0, 1], where $\widetilde{\varphi}(x) = \varphi(x)$, to the whole number line. Assuming that $\widetilde{\varphi}(x)$ is a smooth function, we substitute the relation (9) into the boundary conditions (2). We obtain two equalities: $\widetilde{\varphi}(x) = -\widetilde{\varphi}(-x),\ x \in \mathbf{R}$, i.e., the function $\widetilde{\varphi}(x)$—odd, and

$$\widetilde{\varphi}'(1+x) = 2\widetilde{\varphi}'(x) - \widetilde{\varphi}'(1-x),\ x \in \mathbf{R}, \tag{10}$$

where it is taken into account that $\widetilde{\varphi}'(x)$—is an even function. We integrate the equality (10) over the interval $[0, x]$, and we get

$$\widetilde{\varphi}(1+x) = 2\widetilde{\varphi}(x) + \widetilde{\varphi}(1-x),\ x > 0. \tag{11}$$

The relation (11) allows us to extend the function $\widetilde{\varphi}(x) = \varphi(x),\ x \in [0, 1]$, from the segment [0, 1] to the semiaxis $x > 0$, then we continue the function to the semiaxis $x < 0$ as an odd function.

3 Mixed Problem for an Inhomogeneous Wave Equation with Zero Initial Deviation

Consider the following generalized mixed problem

$$u_{tt}(x,t) = u_{xx}(x,t) + f(x,t), \quad (x,t) \in (0,1) \times (0,+\infty), \tag{12}$$

$$u(0,t) = 0, \quad u_x(0,t) = u_x(1,t), \quad t \geq 0, \tag{13}$$

$$u(x,0) = u_t(x,0) = 0, \quad x \in [0,1], \tag{14}$$

where $f(x,t)$ is a function of class Q.

The formal solution of the problem (12)–(14) by the Fourier method is

$$\begin{aligned} u(x,t) = {} & \tfrac{1}{2}\int\limits_0^t d\tau \int\limits_0^{t-\tau} \Big\{2(x+\eta)(1, f(\xi,\tau)) + \\ & +4\sum_{n=1}^{\infty}\big[(f(\xi,\tau),(1-\xi)\sin 2\pi n\xi)\sin 2\pi n(x+\eta) + \\ & +(f(\xi,\tau),\cos 2\pi n\xi)(x+\eta)\cos 2\pi n(x+\eta)\big] + \\ & +2(x-\eta)(1, f(\xi,\tau)) + 4\sum_{n=1}^{\infty}\big[(f(\xi,\tau),(1-\xi)\sin 2\pi n\xi)\sin 2\pi n(x-\eta) + \\ & +(f(\xi,\tau),\cos 2\pi n\xi)(x-\eta)\cos 2\pi n(x-\eta)\big]\Big\}d\eta, \end{aligned}$$

we used the rule (5) and took the integrals out of the signs of the sums. Let's combine terms with arguments $(x+\eta)$ and $(x-\eta)$, we get

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$$u(x,t) = \tfrac{1}{2}\int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} \{2\eta(1, f(\xi,\tau)) + \\ +4\sum_{n=1}^{\infty}\big[(f(\xi,\tau), (1-\xi)\sin 2\pi n\xi)\sin 2\pi n\eta + \\ +(f(\xi,\tau), \cos 2\pi n\xi)\eta\cos 2\pi n\eta\big]\}d\eta = \tfrac{1}{2}\int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} \widetilde{f}(\eta,\tau)d\eta, \tag{15}$$

the last equality is explained by the fact that the bracketed expression $\{\cdot\}$ in (15), as it follows from the formula (6), has the "sum" $\widetilde{f}(\eta,\tau)$, where $\widetilde{f}(\eta,\tau)$ is the extension of the function $f(\eta,\tau)$ along τ to the entire real axis using the same formulas, which is for the function $\varphi(x)$.

Thus, fair

Theorem 4 *The solution $u(x,t)$ of the generalized mixed problem (12)–(14) is a function of class Q defined by the formula*

$$u(x,t) = \frac{1}{2}\int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} \widetilde{f}(\eta,\tau)d\eta. \tag{16}$$

From the formula (16), using the continuation formulas, we obtain the estimate

$$\|u(x,t)\|_{\mathcal{L}(Q_T)} \le c_T\|f(x,t)\|_{\mathcal{L}(Q_T)}, \quad \forall T > 0,\ c_T = \text{const} > 0,$$

this confirms that $u(x,t)$ is a function of class Q.

4 A Mixed Problem for an Inhomogeneous Wave Equation with a Nonzero Initial Deviation

Consider a generalized mixed problem

$$u_{tt}(x,t) = u_{xx}(x,t) + f(x,t), \quad (x,t) \in (0,1)\times(0,+\infty), \tag{17}$$

$$u(0,t) = 0, \quad u_x(0,t) = u_x(1,t), \quad t \ge 0, \tag{18}$$

$$u(x,0) = \varphi(x),\ u_t(x,0) = 0, \quad x \in [0,1], \tag{19}$$

where $f(x,t)$ is a function of class Q, $\varphi(x) \in \mathcal{L}(0,1)$.

The formal solution of the problem (17)–(19) by the Fourier method is $u(x,t) = u_0(x,t) + u_1(x,t)$, where $u_0(x,t)$ is the series (4) and $u_1(x,t)$ is the series (15). Therefore, based on Sects. 2 and 3, we get

Theorem 5 *Generalized mixed problem (17)–(19) has a solution $u(x,t)$ of class Q defined by the formula*

$$u(x,t)=\frac{1}{2}[\widetilde{\varphi}(x+t)+\widetilde{\varphi}(x-t)]+\frac{1}{2}\int\limits_0^t d\tau \int\limits_{x-t+\tau}^{x+t-\tau}\widetilde{f}(\eta,\tau)d\eta. \tag{20}$$

5 Mixed Problem for the Telegraph Equation

We use the results of Sects. 2, 3 and 4 to solve the following problem:

$$u_{tt}(x,t)=u_{xx}(x,t)-q(x,t)u(x,t), \quad (x,t)\in(0,1)\times(0,+\infty), \tag{21}$$

$$u(0,t)=0, \quad u_x(0,t)=u_x(1,t), \quad t\geq 0, \tag{22}$$

$$u(x,0)=\varphi(x), \; u_t(x,0)=0, \quad x\in[0,1], \tag{23}$$

where $\varphi(x)\in\mathcal{L}(0,1)$, the function $q(x,t)$ is such that there is a function $q_0(x)\in\mathcal{L}(0,1)$, such that $|q(x,t)|\leq q_0(x)$, the function $q(x,t)u(x,t)$ is a function of class Q.

From Theorem 5 we obtain that finding a solution to the problem (21)–(23) in the class Q reduces to finding in this class the solution of the integral equation

$$u(x,t)=\frac{1}{2}[\widetilde{\varphi}(x+t)+\widetilde{\varphi}(x-t)]-\frac{1}{2}\int\limits_0^t d\tau \int\limits_{x-t+\tau}^{x+t-\tau} \widetilde{q(\eta,\tau)u}(\eta,\tau)d\eta, \tag{24}$$

where $\widetilde{q(\eta,\tau)u}(\eta,\tau)$ is the extension along η to the entire real axis from the interval $[0,1]$ for each τ of the function $q(\eta,\tau)u(\eta,\tau)$ by the same formulas as the function $\varphi(x)$.

The integral equation has a unique solution in the class Q obtained by the method of successive substitutions. This solution is given by the formula

$$u(x,t)=A(x,t)=\sum_{n=0}^{\infty}a_n(x,t), \tag{25}$$

where

$$a_0(x,t)=\tfrac{1}{2}[\widetilde{\varphi}(x+t)+\widetilde{\varphi}(x-t)],$$
$$a_n(x,t)=\tfrac{1}{2}\int\limits_0^t d\tau \int\limits_{x-t+\tau}^{x+t-\tau}\widetilde{f}_{n-1}(\eta,\tau)d\eta, \quad n=1,2,\ldots,$$

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where $\widetilde{f}_n(\eta,\tau) = f_n(\eta,\tau) = -q(\eta,\tau)a_n(\eta,\tau)$ for $\eta \in [0,1]$, $n = 0, 1, \ldots$, $f_n(\eta,\tau)$ extends over the variable η from $[0, 1]$ to the whole line in the same way as the function $\varphi(x)$, $\widetilde{f}_n(\eta,\tau) = -\widetilde{q(\eta,\tau)a_n}(\eta,\tau)$.

The formula (25) can be called the generalized d'Alembert formula.

Theorem 6 *If $\varphi(x) \in \mathcal{L}(0,1)$ then the $A(x,t)$ (25) converges absolutely and uniformly (with exponential speed) in the rectangle Q_T for any $T > 0$.*

The proof of the theorem follows directly from the following estimate for the common term of the series (25).

Lemma 1 *Let $\varphi(x) \in \mathcal{L}(0,1)$, T—arbitrary positive number. Then the estimates hold*

$$\|a_n(x,t)\|_{C(Q_T)} \le c_T^{n+1}\|q_0\|_1^n\|\varphi\|_1\frac{T^{n-1}}{(n-1)!},\quad n \in \mathbf{N},\ c_T = \text{const} > 0.$$

The proof of the lemma is carried out using the method of mathematical induction.

6 The Term-by-Term Integration Theorem

Here we justify Theorem 2 on the term-by-term integration of the biorthogonal expansion with respect to the system $\{u_k(x)\}$ integrable on the interval $[0, 1]$ functions. We adhere to the well-known scheme of proving the Lebesgue theorem, with the correction that now the expansion in a series is not carried out according to orthonormal system, but biorthogonal system. Let us rename $\varphi(x)$ in Theorem 2 by $f(x)$.

So, let a function $f(x) \in \mathcal{L}(0,1)$ be given, which has as its biorthogonal expansion in the $\{u_k(x), v_k(x)\}$ system

$$\begin{aligned} &2x(1, f) + 4\sum_{n=1}^{\infty}[(f(\tau), (1-\tau)\sin 2\pi n\tau)\sin 2\pi nx + \\ &+(f(\tau), \cos 2\pi n\tau)x\cos 2\pi nx]. \end{aligned} \tag{26}$$

Let $[A, B] \subseteq [0, 1]$, then it is required to prove that

$$\begin{aligned} &\int_A^B f(x)dx = 2\int_A^B x(1, f)dx + 4\sum_{n=1}^{\infty}\int_A^B\big[(f(\tau), (1-\tau)\sin 2\pi n\tau)\sin 2\pi nx + \\ &+(f(\tau), \cos 2\pi n\tau)x\cos 2\pi nx\big]dx, \end{aligned}$$

those, the series (26) can be integrated term by term, the resulting series converges and its sum is equal to $\int_A^B f(x)dx$. In this case, the series itself (26) may diverge.

Consider the function

$$\varphi(x) = \begin{cases} 1, & x \in [A, B], \\ 0, & x \in [0, 1] \setminus [A, B]. \end{cases}$$

Each of the systems $\{u_k(x)\}, \{v_k(x)\}$, forms an unconditional basis in the space $\mathcal{L}^2(0, 1)$. Let us expand the function $\varphi(x)$ into a series in the system $\{v_k(x), u_k(x)\}$, and call it the conjugate series:

$$\begin{aligned} \varphi(x) &\sim 2\alpha_0 + 4\sum_{k=1}^{\infty}[\alpha_k(1-x)\sin 2\pi kx + \beta_k \cos 2\pi kx] = \\ &= 2(\varphi(\tau), \tau) + 4\sum_{k=1}^{\infty}[(\varphi(\tau), \sin 2\pi k\tau)(1-x)\, sin2\pi kx + \\ &+ (\varphi(\tau), \tau\cos 2\pi k\tau)\cos 2\pi kx]. \end{aligned} \tag{27}$$

Let us calculate the coefficients $\alpha_0, \alpha_k, \beta_k,\ k \geq 1$, of the series (27). We have

$$\alpha_0 = (\varphi(\tau), \tau) = \int_A^B \tau d\tau = \tfrac{1}{2}(B^2 - A^2),$$

$$\alpha_k = (\varphi(\tau), \sin 2\pi k\tau) = \int_A^B \sin 2\pi k\tau d\tau = \tfrac{1}{2\pi k}(\cos 2\pi kA - \cos 2\pi kB),$$

$$\beta_k = (\varphi(\tau), \tau\cos 2\pi k\tau) = \int_A^B \tau\cos 2\pi k\tau d\tau = \tfrac{1}{2\pi k}[B\sin 2\pi kB - A\sin 2\pi kA +$$
$$+ \tfrac{1}{2\pi k}(\cos 2\pi kA - \cos 2\pi kB)].$$

Let us substitute the obtained relations for the coefficients into the partial sum $S_n(x)$ of the series (27):

$$\begin{aligned} S_n(x) &= B^2 - A^2 + 4\sum_{k=1}^{n}\Big[\tfrac{1}{2\pi k}(\cos 2\pi kA - \cos 2\pi kB)(1-x)\sin 2\pi kx + \\ &+ \tfrac{1}{2\pi k}(B\sin 2\pi kB - A\sin 2\pi kA)\cos 2\pi kx + \tfrac{1}{4\pi^2k^2}(\cos 2\pi kA - \\ &- \cos 2\pi kB)\cos 2\pi kx\Big]. \end{aligned}$$

Let us prove that (1) the sequence $\{S_n(x)\}$ converges $\forall x \in [0, 1]$, (2) the sequence $\{S_n(x)\}$ is uniformly bounded in n and x to $[0, 1]$.

(1). To prove the convergence of the series (27), we apply the Dirichlet-Abel test and the comparison test for numerical series. We transform the products of trigonometric functions into sums and group terms. We will receive

$$S_n(x) = B^2 - A^2 + \tfrac{1-x-A}{\pi} \sum_{k=1}^{n} \tfrac{\sin 2\pi k(A+x)}{k} - \tfrac{1-x+A}{\pi} \sum_{k=1}^{n} \tfrac{\sin 2\pi k(A-x)}{k} + \\ + \tfrac{x-1+B}{\pi} \sum_{k=1}^{n} \tfrac{\sin 2\pi k(B+x)}{k} + \tfrac{1-x+B}{\pi} \sum_{k=1}^{n} \tfrac{\sin 2\pi k(B-x)}{k} + \\ + \tfrac{1}{\pi^2} \sum_{k=1}^{n} \tfrac{1}{k^2} (\cos 2\pi k B - \cos 2\pi k A) \cos 2\pi k x. \tag{28}$$

According to the usual scheme, we obtain the estimates

$$\Big|\sum_{k=1}^{n} \sin 2\pi k(A \pm x)\Big| \le \frac{1}{|\sin \pi(A \pm x)|}, \quad \forall n, \ \forall x \in [0, 1],$$

$A \pm x \neq 0$, $A + x \neq 1$. If $A \pm x = 0$ or $A + x = 1$, then the corresponding sums are equal to zero;

$$\Big|\sum_{k=1}^{n} \sin 2\pi k(B \pm x)\Big| \le \frac{1}{|\sin \pi(B \pm x)|}, \quad \forall n, \ \forall x \in [0, 1],$$

$B - x \neq 0$, $B \pm x \neq 1, 2$. If $B \pm x = 1, 2$ or $B - x = 0$, then the corresponding sums are equal to zero.

Thus, the sums of sines in the first four partial sums in (28) are bounded in absolute value for all values of n and $x \in [0, 1]$. Consequently, the series corresponding to these sums converge in every point $x \in [0, 1]$. The series corresponding to the last sum in (28) converges absolutely and uniformly on the set $[0, 1]$.

Thus, the sequence $\{S_n(x)\}$ converges at every point $x \in [0, 1]$, i.e. the series (27) converges on $[0, 1]$.

(2). Let us prove that there is a constant $c > 0$ such that $|S_n(x)| \le c$, $\forall n$, $\forall x \in [0, 1]$. To do this, we prove that each of the sums on the right-hand side (28) is uniformly bounded.

Let us use the well-known estimate ([7, p. 318])

$$\Big|\sum_{k=1}^{n} \frac{\sin kt}{k}\Big| \le 2\sqrt{\pi}, \quad \forall n, \ \forall t \in \mathbf{R}.$$

Putting in the first sum in (28) $t = 2\pi(A + x)$, we obtain

$$\Big|\sum_{k=1}^{n} \frac{\sin 2\pi k(A + x)}{k}\Big| = \Big|\sum_{k=1}^{n} \frac{\sin kt}{k}\Big| \le 2\sqrt{\pi}, \quad \forall n, \ \forall x \in [0, 1].$$

Similarly, we evaluate the next three sums in (28). For the last sum in (28), we obtain an upper bound in terms of the constant $c = 4$, $\forall n$, $\forall x \in [0, 1]$, since

$\sum_{k=1}^{n} \frac{1}{k^2} < 2, \ \forall n$. For the sum $S_n(x)$, we obtain an estimate uniform in n and $x \in [0, 1]$ in terms of the constant $c_1 = 1 + \frac{24}{\sqrt{\pi}} + \frac{4}{pi^2}$:

$$|S_n(x)| \leq c_1, \quad \forall n \geq 1, \ \forall x \in [0, 1]. \tag{29}$$

The results obtained in (1), (2) make it possible to apply the Lebesgue theorem on passing to the limit ([7, p. 139]):

$$\int_0^1 f(x)\varphi(x)dx = \lim_{n\to\infty} \int_0^1 f(x)S_n(x)dx,$$

or, use the relation (27),

$$\int_A^B f(x)dx = 2\alpha_0 \int_0^1 f(x)dx + 4 \sum_{k=1}^{\infty} \Big[\alpha_k \int_0^1 f(x)(1-x)\sin 2\pi kx dx +$$
$$+\beta_k \int_0^1 f(x)\cos 2\pi kx dx\Big] = 2(1, f) \int_A^B x dx +$$
$$+4 \sum_{k=1}^{\infty} \Big[(f(\tau), (1-\tau)\sin 2\pi k\tau) \int_A^B \sin 2\pi kx dx +$$
$$+(f(\tau), \cos 2\pi k\tau) \int_A^B x \cos 2\pi kx dx\Big],$$

those, we get the required formula. Theorem 2 is proved.

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