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Exact solutions to fall of particle to singular potential: classical versus quantum cases

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Exact solutions describing a fall of a particle to the centre of a non-regularized singular potential in classical and quantum cases are obtained and compared. We inspect the quantum problem with the help of conventional Schrödinger's equation. During the fall, the wave function spatial localization area contracts into a single zero-dimensional point. For the fall-admitting potentials, the Hamiltonian is non-Hermitian. Because of that, the wave function norm occurs time-depended. It demands an extension to this case of the continuity equation and rules for mean value calculations. Surprisingly, the quantum and classical solutions exhibit striking similarities. In particular, both are self-similar at the particle energy equals zero. The characteristic spatial scales of the quantum and classical self-similar solutions obey the same temporal dependence. We present arguments indicating that these self-similar solutions are attractors to a broader class of solutions, describing the fall at finite energy of the particle.

1. Introduction

Collapses, i.e. spatio-temporal evolutions of smooth solutions resulting in the formation of singularities, happen in various physical phenomena and play a significant role there. Suffice to mention implosion of spherical and cylindrical shock waves [1–3]; the collapse of bubbles in a liquid [3–6]; the self-focusing in nonlinear optics [7–9]; gradient catastrophe of acoustic waves [10]; Langmuir's collapse in plasma physics [11]. For more examples and their discussions, see, e.g. review [12] and references therein.

Among various collapses, a fall of a quantum particle to the centre of a singular spherically symmetric potential, also known as *quantum collapse*, has a special significance: the potentials admitting the quantum collapse make the Hamiltonian non-Hermitian; see below, §4. As a result, Schrödinger's equation with this potential fails to produce the ground state [13,14]. This gives rise to very unusual properties of wave functions so that the conventional rules for the mean value calculations and continuity equation cannot be used in this case and demand reconsideration; see below.

In 2023, revising fundamental concepts of quantum mechanics might seem peculiar, to say the least. Therefore, we want to clarify that the suggested changes have no bearing on the common problems of non-relativistic quantum mechanics. They only affect the particular solutions to Schrödinger's equation with the non-Hermitian Hamiltonian to overcome the intrinsic contradictions of the conventional rules arising in this case.

The quantum collapse is a rare but not the only case of non-Hermitian Hamiltonians in quantum mechanics. They may also be introduced in other essentially time-depended problems, e.g. in the α-decay, where a complex value of energy corresponds to the decaying in time probability to find the particle in a quasi-discrete level [15] or, in more general terms, in various manifestations of resonant scattering of particles by potentials with quasi-discrete levels, known as Fano resonances [16] etc.

In contrast to conventional Hermitian Hamiltonians, each case of Schrödinger's equation with a non-Hermitian Hamiltonian requires an individual consideration valid, generally speaking, only for a given problem. Accordingly, the approach of the present paper is explicitly developed for the fall to the centre. However, the method employed here is much broader and may also be applied to other problems, whose examples are mentioned above.

Study of Schrödinger's equation with collapse-admitting potentials has a long-lasting history [17–37]. Its results are discussed in reviews [12,38] and enter text-books [13,14]. Nonetheless, most of them are based on various regularization procedures (cutoff of a singular potential at the vicinity of the singularity, a shift of the boundary conditions from the singular point to its proximity, incorporation of nonlinear terms etc.). On the other hand, any regularization procedure implies (explicitly or implicitly) that, at vanishing regularization parameters, the regularized solutions converge to non-regularized ones. This is not the case for the quantum collapse: at the point of the potential singularity, the wave functions do not have any definite limit [13,14]. Therefore, a continuous transition from a regularized problem to its original non-regularized version becomes impossible. Thus, the fundamental question of whether a spatial localization area for a wave function obeying Schrödinger's equation indeed can collapse to a zero-dimensional point remains open.

The argument that, close to the singularity, the collapse conditions usually are violated does not compromise the problem. The issue is common to all collapses. Its resolution is well known: the collapse-admitting problem describes the most physically-significant part of the dynamics, lasting as long as the spatial scale of the solution remains larger than the one where the collapse-breaking terms become essential. At the same time, the problem description in terms of the collapse-admitting approach is more simple, informative, and convenient, that those based on the incorporation of the collapse-breaking corrections [12].

Therefore, here we *intentionally* avoid any regularization. The questions we answer in this study are whether there are any *exact* collapse-exhibiting solutions to Schrödinger's equation valid in all space and, if so, whether they agree with the fundamentals of quantum mechanics, despite the non-Hermitian Hamiltonian.

We obtain a set of such solutions. However, they are too unusual to allow straightforward interpretations. The main goal of the publication is to draw the community's attention to the *interpretations* of the solutions rather than the solutions themselves. We did our best to interpret them, but we do not claim the given interpretation is ultimate···

The paper has the following structure. In §2, we discuss the classical problem. In §3, we inspect a specific example of this case: a fall of an electron to a heavy dipole and reveal the role of radiative losses. In §4, we formulate the quantum problem. In §5, we derive the applicability condition for applying the quasi-classical approximation to the problem in question. In §6, we obtain a family of, exhibiting collapse, exact solutions to Schrödinger's equation. In §7, we inspect the associated with these solutions peculiarities of time-depended norms. Section 8 is devoted to calculations of mean values with the help of wave functions with time-dependent norms. In §9, we describe and discuss the modification of the continuity equation required in this case. In §10, we preset and inspect specific examples of the solutions to the quantum problem. Section 11 is devoted to conclusions. Appendix contains some ancillary calculations.

2. Classical problem

To begin with, we consider classical collapse. Let us recall its main features [39]. The spherical symmetry of the problem results in the conservation of the particle angular momentum $\mathbf{M} = \mathbf{r} \times \mathbf{p}$. It means that $\mathbf{r}(t)$ is always perpendicular to the constant vector $\mathbf{M}$, i.e. the particle trajectory is a two-dimensional curve. Then, it is convenient to describe this motion in a polar coordinate system with the origin at the centre of the potential $U(r)$. Next, the angular momentum conservation leads to the following relation:

$$\dot{\varphi} = \frac{M}{mr^2}, \tag{2.1}$$

where dot stands for $\mathrm{d}/\mathrm{d}t$, and m is the particle mass.

The energy integral has the form

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + U(r) \equiv \frac{m\dot{r}^2}{2} + U_{\text{eff}}(r). \tag{2.2}$$

Here

$$U_{\text{eff}}(r) = U(r) + \frac{mr^2\dot{\varphi}^2}{2} \equiv U(r) + \frac{M^2}{2mr^2}, \tag{2.3}$$

see equation (2.1).

The introduced $U_{\text{eff}}(r)$ makes it possible to exclude φ from the energy integral. The obtained equation is one-dimensional and readily integrated for any $U(r)$. The result is as follows:

$$\begin{aligned} t - t_{\text{ini}} &= \pm \int_{r_{\text{ini}}}^{r} \frac{\mathrm{d}r'}{\sqrt{(2/m)[E - U_{\text{eff}}(r')]}} \\ &\equiv \pm \int_{r_{\text{ini}}}^{r} \frac{\mathrm{d}r'}{\sqrt{(2/m)[E - U(r')] - (M^2/m^2r'^2)}}; \end{aligned} \tag{2.4}$$

where E stands for the particle energy, and $r_{\text{ini}} \equiv r(t_{\text{ini}})$ is the initial condition. Note that we keep two signs of the square root in equation (2.4). It means that the solution equation (2.4) has two branches, so that both $t - t_{\text{ini}}$ and $\mathrm{d}r/\mathrm{d}t$ may have any sign.

This fact is essential. To understand that, consider a finite motion of the particle corresponding to nonlinear oscillations of r between $r_{\min}$ and $r_{\max}$, where $r_{\min,\max}$ are two sequential roots of the equation $E = U_{\text{eff}}(r)$, and $r_{\min} < r_{\max}$. The following expression gives the period of the oscillations:

$$T = 2\int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{\sqrt{(2/m)[E - U_{\text{eff}}(r)]}}. \tag{2.5}$$

When $r(t)$ increases, $\mathrm{d}r/\mathrm{d}t > 0$. It corresponds to the sign plus in equation (2.4). By contrast, the stage of the motion when $r(t)$ decreases corresponds to the sign minus. The conclusion is that at the return points, namely at $r = r_{\min}$ and $r = r_{\max}$, a transition between the solution branches takes place. Since for both branches $\mathrm{d}r/\mathrm{d}t = 0$ at $r = r_{\min,\max}$, see equation (2.4), the transition does not violate smoothness of the dependence $r(t)$. Thus, every time, when r reaches a given value with the same sign of $\mathrm{d}r/\mathrm{d}t$, t_{ini} in equation (2.4) increases by the period of oscillations T. Bearing it in mind, we proceed with the discussion of the collapse.

The expression under the square root in equation (2.4) must be non-negative. Then, the point $r = 0$ is accessible, i.e. collapse is possible, provided

$$[r^2 U(r)]_{r\to 0} \leq -\frac{M^2}{(2m)}. \tag{2.6}$$

For definiteness, in what follows, we consider the potential

$$U(r) = -\beta/r^2; \quad (\beta > 0), \tag{2.7}$$

Notably, that potential equation (2.7) does exist in nature. For example, it describes the interaction of a point electric charge with a particle possessing zero total electric charge but a finite dipole moment [35]; see below, §3 (the issue is essential in understanding electron capture by a polar molecule [20]). It also arises in many other physical problems [40], such as certain quantum three-body problems [21,41], the physics of cold atoms [42,43], polymer physics [44], the near-horizon problem for certain black holes [45] etc. Thus, the collapse in this potential has practical importance in various branches of physics.

The application of condition equation (2.6) to potential equation (2.7) gives rise to the following inequality: $\beta \geq M^2/2m$. Note that, at $\beta < M^2/2m$, $U_{\mathrm{eff}}(r) > 0$, i.e. the effective potential becomes repulsive. Only a motion with $E > 0$ can occur in this case. In agreement with the mentioned above, for this motion, r is bounded from below by $r = r_{\min}$, satisfying the equality $U_{\mathrm{eff}}(r_{\min}) = E$. It explains why the violation of equation (2.6) makes collapse impossible.

The case $\beta = M^2/2m$ is trivial since the dynamic equation, in this case, is the same as that for a free motion of the particle, when $m\dot{r}^2/2 = E$. Therefore, in what follows, we suppose the strict inequality

$$\beta > \frac{M^2}{2m}. \tag{2.8}$$

First, we consider equations (2.4), (2.7) with a finite negative E. Then, equation (2.4) yields

$$r = \sqrt{-\chi t\left(1 + \frac{t}{T}\right)}, \quad -\frac{T}{2} \leq t \leq 0 \tag{2.9}$$

and

$$r = \sqrt{\chi t\left(1 - \frac{t}{T}\right)}, \quad 0 \leq t \leq \frac{T}{2}. \tag{2.10}$$

where

$$\chi \equiv \frac{2\sqrt{2m\beta - M^2}}{m}, \quad T = \frac{m\chi}{2|E|}. \tag{2.11}$$

(we set $t_{\mathrm{ini}} = 0$ and employ the initial condition $r(0) = 0$). Note that the dimension of χ is length^2/time.

Branch equation (2.9) of the obtained solution describes a fall of the particle to the centre, which begins with $r = r_{\max} \equiv \sqrt{\chi T}/2$, at $t = -T/2$, and ends at the origin of the coordinate system, at $t = 0$. At $t > 0$, the collapse terns into escape, described by branch equation (2.10), so that $r(t)$ increases from 0, at $t = 0$, to $r_{\max}$, at $t = T/2$.

Note also that T and $r_{\max}$ are the only problem constants with the dimensions of time and length, respectively. When, during the collapse, $r(t)$ becomes much smaller than $r_{\max}$, the latter ceases to play the role of the problem characteristic scale. Moreover, the condition $r(t) \ll r_{\max}$ holds at $|t| \ll T$, see equations (2.9), (2.10). It means that, in this region, T does not determine the characteristic temporal scale too. That is to say, close to the completion of the collapse (beginning of the escape), the problem loses its characteristic scales both in time and space. Then, according to the general principles of dimensional analysis [46,47] the problem must become self-similar, when its dynamic is described by a dimensionless ratio of r to a certain power of t, instead of the two independent variables r and t, in non-self-similar cases; see also below §4.

For the problem in question, it is so indeed: in the specified region, the term t/T in equations (2.9), (2.10) may be dropped. It transforms both branches into the self-similar solution

$$\frac{r}{\sqrt{\chi|t|}} = 1, \tag{2.12}$$

where the dimensionless $\xi = r/\sqrt{\chi|t|}$ may be regarded as a new self-similar quantity.

What happens if E tends to zero from below? In this limit, both T and $r_{\max}$ tend to infinity, and the periodic nonlinear oscillations of the particle become an aperiodic motion when, at $t<0$, the particle falls to the centre from infinity and then, at $t>0$ escapes from the centre, returning to infinity.

Remarkably, that in this case, the self-similar solution equation (2.12) becomes *exact*. At the same time, any general-type solution with finite $E<0$ is transformed into the self-similar one at the late stage of the collapse (the initial stage of the escape); see equations (2.9), (2.10). In other words, the exact self-similar solution equation (2.12), valid at $E=0$, is an *attractor* for any other solution with a finite E exhibiting the collapse (escape). For this reason, the case $E=0$ will play a special role in the proceeding discussion; see §§4–10.

It is important to stress that close to the moment of the collapse completion or escape beginning, the transformation of a solution with any finite value of E into a self-similar form is not a specific feature of the potential equation (2.7). It is a generic property of any collapse-admitting potential with a power-type singularity. Indeed, consider a potential with the singularity $\sim 1/r^s$. To satisfy the collapse condition equation (2.6) at $M \neq 0$, we must have $s \geq 2$. However, if $M=0$, the only restriction is $s>0$. In this case, even the Coulomb potential $\sim 1/r$ is collapse-admitting. Either way, at $r \to 0$ and *any* finite E, the term $\sim 1/r^s$ makes the overwhelming contribution to the square root in equation (2.4). Then, dropping other terms under the square root sign and evaluating the integral, we obtain a universal self-similar solution in the form $r/|t|^{2/(2+s)} = \text{const}$, whose particular case at $s=2$ is equation (2.12). In other words, at $r \to 0$, the dependence $r = \text{const}|t|^{2/(2+s)}$ is the only universal asymptotic to *any* collapse(escape)-exhibiting solution to the problem in question.

Returning to the potential equation (2.7), it is relevant to calculate the radial component of the particle momentum $p_r = m\,\mathrm{d}r/\mathrm{d}t$. At $E=0$, it reads:

$$p_r = \pm \frac{\chi m}{2\sqrt{\chi|t|}}, \tag{2.13}$$

where the signs minus and plus correspond to the collapse ($t<0$) and escape ($t>0$), respectively. Note that p_r diverges at $t \to 0$.

Taking into account the periodicity of the partile motion, we can rewrite the solution equation (2.9), (2.10) in an equivalent form, which sometimes is more convenient for analysis. To this end, we employ instead of equation (2.9), the same branch shifted by period T along the t-axis. It can be done with the help of the formal transformation $t \to t-T$ in equation (2.9). It is easy to see that this procedure converts equation (2.9) into equation (2.10). At the same time, the validity domain of the converted branch becomes $T/2 \leq t \leq T$. Merging it with the validity domain stipulated by equation (2.10), we obtain an equivalent form of a single period of the solution described only by equation (2.10), which, however, now is valid for $0 \leq t \leq T$.

To complete the inspection of the classical problem, we present an explicit expression for $\varphi(t)$. We readily obtain it by integrating equation (2.1). Doing that, it is convenient to use for $r(t)$ the above equivalent form of the solution. It gives rise to the following formula:

$$\varphi(t) = a \ln \frac{t}{T-t}, \quad 0 \leq t \leq T, \tag{2.14}$$

where $a \equiv M/m\chi$. We stress that, for the obtained solution, the domain of the non-trivial values of φ extends from minus to plus infinity. As an example, figure 1 shows the trajectory of the particle at $a=5$.

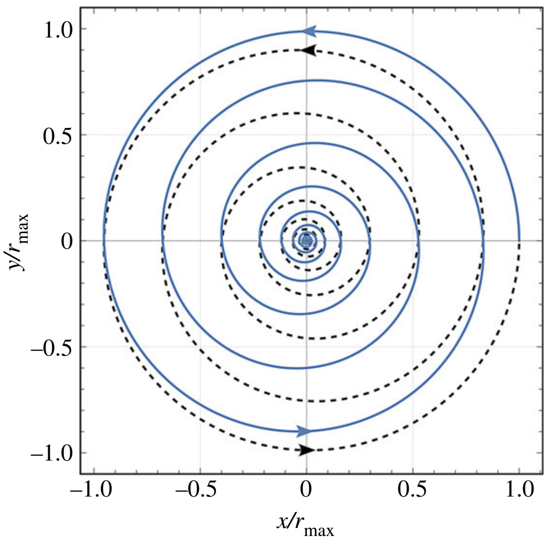

Figure 1. An example of a classical collapse-escape trajectory in a Cartesian coordinate system; $a = 5$. The dashed black and solid blue lines designate the two parts of the trajectory described by equations (2.10), (2.14) with $0 \leq t \leq T$. Here, $0 \leq t \leq T/2$ and $T/2 \leq t \leq T$ correspond to escape and collapse, respectively.

3. Fall of electron to heavy dipole: radiative losses

This section considers a specific example of the classical collapse: a fall of a (quasi)classical electron to a heavy neutral atom (or molecule) with a fixed finite dipole moment **d**.[1] We consider the most straightforward problem formulation, corresponding to the zero angular momentum of the electron. Then, **d** must be directed to the falling electron to minimize the electrostatic interaction energy. Thus, the problem is a particular case of the one discussed above, where $M = 0$. We can neglect the displacement of the atom's centre of mass owing to the tremendous difference in the masses of the electron and atom. However, the falling electron moves with increasing acceleration and hence must emit electromagnetic waves. The emission decreases the electron energy. It may affect the structure of the above solutions. To elucidate this effect, we calculate the radiative losses. We do it based on the above solution (obtained without consideration of the losses) regarding this solution as the zeroth approximation. Then, the power emitted by a non-relativistic electron moving with the acceleration w is given by Larmor's formula [48]

$$I = \frac{2e^2 w^2}{3c^3}, \tag{3.1}$$

where e is the electron charge, and c stands for the speed of light in a vacuum.

For the problem in question, the closer collapse completion moment, the larger w. Therefore, it is natural to suppose that the effect of the radiative losses increases, as the collapse completion approaches. On the other hand, as shown above, close to the completion moment, the general solution equations (2.9), (2.10) is transformed into the self-similar form equation (2.12). Therefore, it is sufficient to study the radiative losses for $r(t) = \sqrt{-\chi t}$ at $t \leq 0$. In this case, the total energy $E_{\rm rad}$ emitted prior to a given moment t is

$$E_{\rm rad} = \int^t I(t')\,{\rm d}t' = \frac{4e^2\beta^2}{3c^3 m^2 \chi(-\chi t)^2} \equiv \frac{4e^2\beta^2}{3c^3 m^2 \chi r^4}. \tag{3.2}$$

[1]The Hamiltonian of an atom is invariant against the inversion transformation, while the Hamiltonian of a molecule is not. For this reason, a molecule may have a fixed dipole moment in a stationary state. By contrast, the dipole moment of an atom, in a generic case, is zero. Nonetheless, if certain special conditions hold, an atom also may have a finite dipole moment [13].

The upper limit of the integral in equation (3.2) makes the main contribution to its value. Then, it is possible to extend the lower limit to minus infinity, despite the self-similar asymptotic equation (2.12) of the general solution equation (2.9) is valid only in the vicinity of the collapse moment.

To estimate how these losses affect the collapse dynamic, note that the latter is described by equations (2.4) and that close to the collapse completion moment $|U(r)| \gg |E|$. It means that we have to compare E_{rad} not to E but to $U(r)$. In other words, the radiative losses impact on the collapse dynamic becomes substantial at the value of $r \sim r_{\mathrm{rad}}$, where r_{rad} is defined by the condition $|U(r_{\mathrm{rad}})| = E_{\mathrm{rad}}(r_{\mathrm{rad}})$. In contrast, the losses may be neglected at $r \gg r_{\mathrm{rad}}$. Simple algebra results in the following expression for r_{rad}:

$$r_{\mathrm{rad}} = \frac{2|e|}{mc^{3/2}}\sqrt{\frac{\beta}{3\chi}}. \tag{3.3}$$

Let us estimate the order of magnitude of r_{rad}. At $M = 0$, $\chi = 2\sqrt{2\beta/m}$; see equation (2.11). Next, for the dipole moment produced by a polarization of spatial distribution of the valence electrons, β is estimated as $e^2 r_B$, where $r_B = \hbar^2/me^2$ is Bohr's radius. In this case the estimate of the r.h.s. of equation (3.3) reads

$$r_{\mathrm{rad}} \sim \frac{1}{(mr_B)^{3/4}}\left(\frac{|e|}{c}\right)^{3/2} r_B = \alpha^{3/2} r_B, \tag{3.4}$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine-structure constant. The numerical value of $\alpha^{3/2} \approx 6 \times 10^{-4}$. That is to say, $r_{\mathrm{rad}} \ll r_B$.

Since Bohr's radius is the characteristic quantum scale for atomic phenomena, the obtained estimate of r_{rad} means that long before the impact of the radiative losses on the collapse dynamics becomes noticeable, the classical description must be replaced by the corresponding quantum one. This essentially quantum description of the collapse is given below.

4. Quantum problem formulation

Conventionally, the Hamiltonian in Schrödinger's equation is a Hermitian operator with a complete set of orthogonal eigenfunctions. Accordingly, solutions to Schrödinger's equation can be built as eigenfunction expansions. However, in the case of potential equation (2.7), the eigenfunctions corresponding to different E values are not necessarily orthogonal.[2] It is related to their behaviour at $r \to 0$ [14]. Since the orthogonality of Hamiltonian eigenfunctions with different values of E is a direct consequence of its self-adjointness (see, e.g. [13]), the non-orthogonality, even for a single pair of them, means that the Hamiltonian with potential equation (2.7) *is not Hermitian*. It makes the possibility of building general solutions to the corresponding Schrödinger's equation in the form of eigenfunction expansions questionable. At least, the author is unaware of any success in this way.

Regarding particular solutions, which may be built from the set of the Hamiltonian eigenfunctions discussed in the monograph by Morse & Feshbach [14], they do not help much (if any) to understand the collapse dynamics. The point is that in Schrödinger's equation, Hamiltonian eigenfunctions are wave functions of *stationary states*. In the collapse case, building a solution in the form of eigenfunction expansion is an attempt to describe an essentially time-depended process with the help of stationary-state eigenfunctions. It looks like trying to force a

[2]The orthogonality may be imposed as an *additional* condition. It singles out a certain subset of eigenfunctions, while the eigenfunctions that do not satisfy the imposed condition are not orthogonal to the ones belonging to the subset. For details, see [14].

square peg into a round hole. To overcome this difficulty, we consider a spatio-temporal evolution of a wave packet,[3] described by time-depended Schrödinger's equation

$$i\hbar\frac{\partial\Psi}{\partial t}=\hat{H}\Psi;\quad \hat{H}\equiv-\frac{\hbar^2}{2m}\Delta+U(r). \tag{4.1}$$

Here, Δ stands for the Laplacian, and $U(r)$ is given by equation (2.7). Equation (4.1) should be supplemented by the initial condition $\Psi=\Psi_0(\mathbf{r})$ and the one stipulating finiteness of the wave function norm, i.e, convergence of $\int|\Psi|^2d^3\mathbf{r}$.

We employ the power of dimensional analysis to obtain collapsing solutions of equation (4.1) The main concepts of this analysis are as follows [46,47]. Arguments of any mathematical function must be dimensionless quantities. The wave function in equation (4.1) depends on the dimensional $\mathbf{r}$ and t. To make them dimensionless, we have to normalize them on constants with the proper dimensions. These constants should be built from the ones entering our problem. There are only three dimensional constants entering equation (4.1), namely $m,\hbar$, and β. Their most general combination has the form: $m^{\zeta_1}\hbar^{\zeta_2}\beta^{\zeta_3}$, where $\zeta_{1,2,3}$ are real constants. It is easy to see that for the β dimension corresponding to equation (2.7) and any values of $\zeta_{1,2,3}$ this product cannot have the dimensions of $\mathbf{r}$ or t.

The only remaining way to obtain the required constants is to find them in the initial condition. Indeed, the initial condition must have a characteristic spatial scale of the wave function localization r_{ini}. Then, e.g. the combination $mr_{\mathrm{ini}}^2/\hbar$ may be selected as the characteristic temporal scale. Seemingly, it removes the normalization problem. It does not! The point is that if the collapse indeed takes place, the region of the wave function spatial localization contracts. Its size eventually turns to zero. Then, the finite scale associated with the initial conditions ceases to play the role of a characteristic scale of the problem and becomes useless for our purposes.

The other striking conclusion following from the dimensional analysis is that Schrödinger's equation with potential in the form of equation (2.7) cannot have discrete levels but the one with $E=0$ [19]. To see that, once again we consider a more general form of the potential, namely

$$U(r)=-\frac{\beta}{r^s}, \tag{4.2}$$

where the sign of s may be any. At $s<0$ to make expression (4.2) a potential well (not a barrier) β also should be negative. In particular, at $s=1$ equation (4.2) corresponds to the Coulomb field, and for $s=-2$ it is a harmonic oscillator potential. In our case $s=2$. According to precisely the same arguments as those used above in the discussion of the characteristic spatio-temporal scales, we conclude that if a discrete spectrum exists, its levels should be composed as products of powers of $\hbar, m$ and β. There is the only possible product with the required dimension, which gives rise to the following expression for the energy of the levels:

$$E_n=\epsilon_n m\left(\frac{|\beta|m^{s-1}}{\hbar^s}\right)^{2/(2-s)}, \tag{4.3}$$

where ϵ_n is a dimensionless quantity.

It is seen straightforwardly that equation (4.3) coincides with the well-known expressions for the spectrum of a harmonic oscillator and the one in the Coulomb potential. However, at $s=2$ it fails to produce a reasonable result. In this case, the only opportunity to return to physically meaningful application of the obtained expression is to set there ϵ_n to zero.

The above simple arguments result in the conclusion that if the quantum collapse exists, a solution describing its final stage must be self-similar. In this solution, $\mathbf{r}$ varies as a certain power

[3]The idea is not mine. When I was a junior scientist, my adviser Sergei I. Anisimov from Landau Institute told me about this approach to the problem. Moreover, he said that together with Igor E. Dzyaloshinskii, they had found a solution describing the collapse of a wave packet. Many years later, in connection with nanoparticle light scattering, I came across a problem mathematically analogous to the quantum collapse. I recalled this conversation with Anisimov and asked him for details and references. He replied that these results had never been published and details he did not remember. Now, Anisimov and Dzyalashinskii have both passed away. I decided to apply their approach to the problem and make the results available to a broad readership as a small token of my great respect for these distinguished scholars.

of t, and the corresponding dimensionless variable is built as a ratio of $\mathbf{r}$ to this power of t, cf. Equation (2.12) in the classical case. Moreover, the self-similar solution must be an attractor: any solution exhibiting the collapse is transformed into the self-similar form at the final stage of the phenomenon. We have proven that for the classical case; see the convergence of the general solution equations (2.9), (2.10) to the self-similar equation (2.12) in the vicinity of the point $r=0$. We can transfer these classical results to the quantum problem, matching the classical and quantum cases through the quasi-classical approximation, discussed below.

5. Quasi-classical condition

The applicability condition for the quasi-classical approximation implies that the characteristic spatial scale of the wave function variations is much smaller than that for the potential and hence, for the corresponding classical solution. The particle de Broglie wavelength, which is inversely proportional to its momentum, determines the wave function spatial variations. At a given potential, the larger the momentum, the smaller the de Broglie wavelength and, therefore, the more accurate the quasi-classical approximation. For the classical problem formulation, the particle momentum increases with the increase in the coupling constant β. On the other hand, the shape of the potential equation (2.7) is determined by r^2 in the denominator and does not depend on β. Then, it may be expected that the entire dynamic of the quantum particle is quasi-classical, at large enough β.

To check the guess, we must explicitly employ the applicability condition. Due to the introduced U_{eff} the classical particle dynamic becomes one-dimensional, see equations (2.2)–(2.4), and the radial component of the momentum $p_r = m\dot{r}$ plays the role of the corresponding one-dimensional momentum. In this case, the quasi-classical applicability condition reads as follows [13]:

$$\left|\frac{\partial \lambda\!\!\!^{-}}{\partial r}\right| \ll 1, \tag{5.1}$$

where $\lambda\!\!\!^{-} = \hbar/p_r$ and $p_r = \pm\sqrt{2m(E-U_{\mathrm{eff}})}$; see equation (2.2).

Simple calculations transform equation (5.1) into the following expression:

$$\frac{\hbar(2m\beta - M^2)}{|2mEr^2 + 2m\beta - M^2|^{3/2}} \ll 1. \tag{5.2}$$

Thus, at any finite E and $r \to \infty$, the particle motion is quasi-classical. However, we are interested in the opposite limit, namely $r \to 0$. Setting in equation (5.2) r to zero gives rise to the condition

$$\frac{\hbar}{\sqrt{2m\beta - M^2}} \ll 1. \tag{5.3}$$

Equation (5.3) mathematically confirms the guess: at large enough β, the quantum dynamic is always quasi-classical.

6. Exact solution to Schrödinger's equation

Now, when we have unveiled the qualitative features of the quantum problem, we can find its exact self-similar solution. We will look for it in the following form:

$$\Psi = \sum_{\ell} \sum_{m=-\ell}^{\ell} C_{\ell m} \Phi_{\ell}(-\chi t) Y_{\ell}^{m}(\theta,\varphi) R_{\ell}(\xi). \tag{6.1}$$

Here, $C_{\ell m}$ are constants, $Y_{\ell}^{m}(\theta,\varphi)$ stand for the spherical harmonic functions (do not confuse index m with the mass of the particle), $\xi = r/(-\chi t)^{\nu}$; $\Phi_{\ell}(-\chi t)$, $R_{\ell}(\xi)$, χ, ν are yet unknown functions and constants, respectively. Here, the moment $t=0$ corresponds to the complete collapse. Then, the collapse dynamic is described by $t<0$.

We have to stress that, in contrast to the conventional eigenfunction expansion, the ansatz equation (6.1) *is not* a general solution to the problem. Moreover, for the time being, we even cannot say that it is a solution. We *guess* that it is. To ensure that the solution in such a form does exist, we have to find it explicitly. Let us proceed in this way.

It is possible to show (see appendix A) that equation (6.1) may be a self-similar solution to equation (4.1), provided

$$\Phi_\ell(-\chi t) = \mathrm{const}_\ell(-\chi t)^{\mu_\ell}, \tag{6.2}$$

where μ_ℓ are dimensionless constants (generally speaking, complex). Regarding the dimensional constant χ, it is convenient to suppose that $\chi = \hbar/m$. Note that, at this definition, the dimension of χ is the same as that in the classical case, namely length^2/time.

The necessary collapse condition $\ell(\ell+1) < (2m\beta/\hbar^2) - (1/4)$ [13,14], limits the value of ℓ in the sums in equation (6.1). Recalling that the eigenvalues of the square of the angular momentum $\widehat{\mathbf{l}^2}$ are $\hbar^2\ell(\ell+1)$ and denoting them as M^2, we may rewrite the above constraint as

$$\beta > \frac{\hbar^2\ell(\ell+1)}{2m} + \frac{\hbar^2}{8m} \equiv \frac{M^2}{2m} + \frac{\hbar^2}{8m}. \tag{6.3}$$

When $\hbar \to 0$, equation (6.3) coincides with the classical collapse condition; see equation (2.8).

Substituting equations (6.1), (6.2) into equation (4.1) and employing the orthogonality of $Y_\ell^m(\theta,\varphi)$ we obtain a detached equation for R_ℓ. Then, without loss of generality, we consider a single term on the r.h.s. of equation (6.1) as Ψ, dropping the signs of sums, while $C_{\ell m}$ may be dropped owing to the linearity of the problem. Therefore, we can simplify the notations by omitting the subscript ℓ. Eventually, at $\nu = 1/2$, Schrödinger's equation is reduced to the following ordinary differential equation for $R(\xi)$; see appendix A:

$$R'' + \left(\frac{2}{\xi} + i\xi\right)R' + \left(\frac{\gamma}{\xi^2} - 2i\mu\right)R = 0, \tag{6.4}$$

where

$$\gamma \equiv \frac{2m\beta}{\hbar^2} - \ell(\ell+1), \tag{6.5}$$

and prime denotes $\mathrm{d}/\mathrm{d}\xi$.

The authors of [12,49] employed a particular type of solution equations (6.1), (6.2) with $\mu = -(1/2 + i\kappa)$ to study the so-called weak collapse in *nonlinear* Schrödinger's equation. However, in [12,49], a physically meaningful solution exists only at a single value of κ. By contrast, in the linear problem discussed here, the restrictions imposed on μ are much weaker, see below.

Equation (6.4) is exactly integrable. Its general solution is

$$R(\xi) = \frac{1}{\sqrt{\xi}}\left[C_1\xi^{-(i\alpha/2)}{}_1F_1\left(-\frac{1+i\alpha}{4} - \mu; 1 - \frac{i\alpha}{2}; -\frac{i\xi^2}{2}\right)\right.$$
$$\left. + C_2\xi^{i\alpha/2}{}_1F_1\left(-\frac{1-i\alpha}{4} - \mu; 1 + \frac{i\alpha}{2}; -\frac{i\xi^2}{2}\right)\right]. \tag{6.6}$$

Here, $C_{1,2}$ are constants, ${}_1F_1(a;b;z)$ designates the Kummer confluent hypergeometric function of the first kind [50], and

$$\alpha \equiv \sqrt{4\gamma - 1} > 0, \tag{6.7}$$

(do not confuse it with the fine-structure constant!).

The positiveness of the expression under the square root in equation (6.7) follows from the necessary collapse condition; see equations (6.3), (6.5). If the condition does not hold, i.e. $\gamma < 1/4$, α is purely imaginary, and equation (6.4) with purely real α becomes invalid.

Now, when we have introduced the self-similar variable ξ, and obtained the quantum solution equation (6.6), we can rewrite equation (5.3) in an equivalent form, namely

$$\frac{2\chi|t|}{r_{\text{clas}}(t)^2} \ll 1. \tag{6.8}$$

Here, $r_{\text{clas}}(t)$ is the classical self-similar solution equation (2.12). The advantage of this presentation is its physical clarity. Indeed, $\sqrt{-\chi t}$ is the characteristic spatial scale of variations of the obtained self-similar wave function. Then, the quantum particle motion is quasi-classical, provided this scale is small relative to the one for the corresponding classical solution. Notably, while the condition equation (6.8) is valid for any general (i.e. not *necessarily* self-similar) solution, it is written in terms of the self-similar variables. The latter is an additional indication of the importance of self-similar solutions for the problem in question.

Expression (6.6) includes the terms $\xi^{\pm i(\alpha/2)}$, where ξ and α are positive quantities. Then,

$$\begin{aligned}\xi^{\pm i\frac{\alpha}{2}} &= (\mathrm{e}^{i2\pi n+\ln\xi})^{\pm i(\alpha/2)} = \mathrm{e}^{\mp\pi\alpha n}\,\mathrm{e}^{\pm i(\alpha/2)\ln\xi}\\ &= \mathrm{e}^{\mp\pi\alpha n}\left[\cos\left(\frac{\alpha}{2}\ln\xi\right) \pm i\sin\left(\frac{\alpha}{2}\ln\xi\right)\right],\end{aligned} \tag{6.9}$$

where n is an arbitrary integer. Such a singularity is typical for the problem under consideration [13,14].

Expression (6.9) has an infinite number of branches corresponding to different values of n. Every branch's real and imaginary parts have the number of zeros, demonstrating unlimited growth at $\xi \to 0$. For simplicity, in what follows, only the single branch with $n = 0$ is inspected.

According to the definition of ξ, we should emphasize that this quantity diverges at $t = 0$, i.e. at the moment of the collapse completion. Then, while the behaviour of $R(\xi)$ at $\xi \to 0$ is essential from the viewpoint of the solution branching and its other analytical properties, the behaviour of the wave function close to the moment of the collapse completion is practically overwhelmingly determined by the $R(\xi)$ asymptotic at $\xi \to \infty$. This asymptotic is discussed in appendix B.

What is about escape? Schrödinger's equation is invariant against the time-reversal procedure accompanied by the complex conjugation. Then, being applied to equations (6.1), (6.2), (6.6), these transformations generate the wave function describing the particle escape from the centre at $t > 0$. As well as in the classical case, at the moment $t = 0$, the collapse is transformed into escape by transferring from one solution to the other.

7. Time-dependent Norm

We must normalize Ψ to calculate the probability density and the mean values of operators. Conventionally, the norm of a wave function ($||\Psi||$) is a constant. It follows from self-adjointness of $\hat{H}$ [13]. Indeed,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}||\Psi||^2 &\equiv \frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi|\Psi\rangle \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int \Psi^*\Psi\, d^3\mathbf{r}\\ &= \int \Psi\frac{\partial\Psi^*}{\partial t} d^3\mathbf{r} + \int \Psi^*\frac{\partial\Psi}{\partial t} d^3\mathbf{r},\end{aligned} \tag{7.1}$$

where the asterisk stands for the complex conjugation. Bearing in mind equation (4.1), it may be rewritten as

$$\frac{i}{\hbar}\left(\int \Psi\hat{H}^*\Psi^* d^3\mathbf{r} - \int \Psi^*\hat{H}\Psi\, d^3\mathbf{r}\right). \tag{7.2}$$

If $\hat{H}$ is Hermitian, the above expression identically equals zero.[4]

[4] Actually, the case is more subtle. In equations (7.1), (7.2), we integrate over all space at once. A more accurate approach implies that we take the integrals over some finite volume V, and then, consider the limit $V \to \infty$, extending V to all space. For the finite V, the integrals over V can be transformed by Gauss's theorem into integrals over the bounding V surfaces. The corresponding integrals are proportional to the probability density fluxes through these surfaces. The expression equation (7.2) vanishes, provided these fluxes turn to zero at $V \to \infty$ or, in a more general case, the flux from infinity is equal to the one through the origin of the coordinate system. See the discussion of this issue in §9.

In our case, $\hat{H}$ is not Hermitian. Then, the norm may be time-depended. Indeed, simple calculations show that for equations (6.1), (6.2), (6.6) $||\Psi||^2 = C(-\chi t)^{(3/2)+2\mu'}$, where $\mu' = \mathrm{Re}\,\mu$ and C is a constant proportional to $\int_0^\infty |R(\xi)|^2 \xi^2 \, \mathrm{d}\xi$. The convergence of the latter determines the finiteness of C.

Calculations show that a proper choice of the ratio C_2/C_1 in equation (6.6) provides the integral convergence at any values of μ' except $\mu' = -3/4$. At $\mu' = -3/4$ the integral diverges at any $C_{1,2}$; see appendix B. Since $\mu' = -3/4$ is the only value of μ', when $||\Psi||$ would not depend on t (see above), the requirement of the finiteness of $||\Psi||$ *always* makes it time-depended.

Note that the divergence of $||\Psi||^2$ at $\mu' = -3/4$ is logarithmic, i.e. extremely weak. It can be stabilized by any correction making the wave function decay faster than that for the obtained exact solution. The role of this correction can play, e.g. relativistic effects, the radiative losses discussed above, etc. Phenomenologically the stabilization may be introduced as a cutoff of the norm integral at some $\xi_0 \gg 1$. In this case, the norm is a constant, and all further consideration is the same as that in conventional cases with a Hermitian Hamiltonian. Then, we could assume that the case $\mu' = -3/4$ is the only physical one, while all other solutions with $\mu' \neq -3/4$ do not have physical meaning.

However, the cutoff drives the problem beyond its initial, strict all-sufficient formulation of Schrödinger's equation with potential equation (2.7) defined in all unlimited space. It is interesting to see if we can keep the problem formulation and the basic quantum mechanics concepts unmodified for a wave function with a time-dependent norm. We discuss this issue in the next section.

8. probability density and mean value calculations

To be able to employ a wave function with a non-conserved norm we must reexamine several conventional quantum mechanics rules and modify them, if required. The first, arising in this case question, is how to normalize Ψ to obtain the probability density? There are at least two options:

(i) the conventional expression, namely $|\Psi|^2/\mathrm{const}$;
(ii) $|\Psi|^2/||\Psi||^2$.

Both have *pro* and *contra*. Case (i) is conventional. However, in this case, the probability of finding a particle in any point of all space is not equal to unity and varies in time as $(-\chi t)^{(3/2)+2\mu'}$ for collapse and as $(\chi t)^{(3/2)+2\mu'}$ for escape, see the previous section. Then, values of μ' smaller than $-3/4$ are meaningless since they would correspond to the probability of finding the particle larger than unity at $|t| \to 0$. At $\mu' > -3/4$, (i) would mean that the singularity at $r = 0$ acts as a sink for the collapse ($t < 0$) and a source for the escape ($t > 0$). In other words, during the collapse, the particle gradually gets out of our world (where to?), completely disappears at $t = 0$, then, slowly returns at $t > 0$. This scenario sounds rather unusual.

Though case (ii) is also unusual owing to the time-dependence of $||\Psi||$, in this case, the probability of finding a particle in any point of all space identically equals unity, and the problem of communication with the 'other world' does not arise. However, in this case, the normalized wave function does not satisfy Schrödinger's equation (the term $i\hbar\Psi\partial(1/||\Psi||)/\partial t$ remains uncompensated.) On the other hand, since Ψ does, this feature does not contradict to the fundamentals of quantum mechanics. Thus, case (ii) looks more physical.

Nonetheless, the final judgement in favour of either of the two cases must be done with the help of the *uncertainty relations*. The derivation of the latter does not employ the self-adjointness of Hamiltonian and norm conservation (see [13] and appendix C). Therefore, the uncertainty relations must be valid for the problem in question too. Calculating the mean value of $\hat{r}$ and $\hat{p}_r \equiv -i\hbar\partial/\partial r$, and assuming, as usual, that $\Delta r \Delta p_r \sim \langle \hat{r} \rangle \langle \hat{p}_r \rangle$, we readily obtain: in case (i) $\Delta r \Delta p_r \sim \hbar(-\chi t)^{3+4\mu'}$, which does not satisfy the uncertainty relations; in case (ii) $\Delta r \Delta p_r \sim \hbar$, which does. Thus, choice (ii) is correct, while (i) is not.

The angular momentum conservation gives an additional argument supporting the choice of case (ii). Due to the problem symmetry, the mean angular momentum $\langle\widehat{\mathbf{l}^2}\rangle$ must be a conserved quantity. For the given Ψ, we have $\langle\Psi|\widehat{\mathbf{l}^2}|\Psi\rangle = \hbar^2\ell(\ell+1)||\Psi||^2$. To avoid misunderstanding, we stress that here and in what follows, for any operator $\hat{A}$, the expression $\langle\hat{A}\rangle$ designates the mean value of $\hat{A}$, while $\langle\Psi|\hat{A}|\Psi\rangle$ stands for the scalar product of $\langle\Psi|$ and $\hat{A}|\Psi\rangle$, where, generally speaking, the wave functions are not normalized. Therefore, $\langle\hat{A}\rangle = \langle\Psi|\hat{A}|\Psi\rangle/||\Psi||^2$.

Bearing it in mind and taking into account that $||\Psi||$ is time-dependent, we conclude that to make $\langle\widehat{\mathbf{l}^2}\rangle$ conserved, one must get rid of $||\Psi||$ in the expression for $\langle\widehat{\mathbf{l}^2}\rangle$, i.e. choose case (ii). Then, $\langle\widehat{\mathbf{l}^2}\rangle = \hbar^2\ell(\ell+1) = const$, as it should be.

It is also relevant to mention that in case (ii)

$$\langle\hat{r}\rangle = c_r\sqrt{-\chi t};\ \langle\hat{p}_r\rangle = c_p\frac{\hbar}{\sqrt{-\chi t}}, \tag{8.1}$$

where $c_{r,p}$ are dimensionless constants of the order of unity. These expressions are remarkably similar to their classical analogues; see equations (2.12), (2.13).

Regarding the particle energy, note that since the obtained Ψ is not an eigenfunction of $\hat{H}$, only the mean value: $E = \langle\hat{H}\rangle$ makes sense. According to equation (4.1) and (ii),

$$E = \frac{\langle\Psi|\hat{H}|\Psi\rangle}{||\Psi||^2} = \frac{i\hbar\langle\Psi|\partial\Psi/\partial t\rangle}{||\Psi||^2} = \frac{C_E}{t}. \tag{8.2}$$

Here, the constant C_E is given by a certain integral. The integral converges at $\mu' < -3/4$ and $\mu' > -1/4$; see appendix B. On the other hand, the energy conservation law requires that $E = \text{const}$. It is compatible with equation (8.2) only if $C_E = 0$, i.e. $E = 0$. Tedious evaluation of the integral in the complex plane gives rise to the same result; see also the discussion at the end of §4 concerning the non-existence of the discrete spectrum but the level with $E = 0$ for the potential equation (2.7), which also can be applied to this case.

We stress that the condition $E = 0$ also follows from the self-similarity of the obtained solution. This argument is valid both in classical and quantum cases. Indeed, at $E \neq 0$, the problem possesses constants with the dimensions of time and length, which can be built with the help of m, β and $|E|$; see equations (2.11), (2.12). In the quantum case, due to the additional constant $\hbar$, it can be done even in various ways. The existence of a characteristic spatio-temporal scale breaks self-similarity, and expressions equations (2.12), (6.1) fail to be *exact* solutions to the corresponding problems. It does not mean that, at $E \neq 0$, collapsing solutions do not exist at all. They may exist in non-self-similar forms; see equations (2.9), (2.10). However, as we have several times emphasized above, the self-similar solutions remain attractors to non-self-similar ones.

9. Generalized continuity equation

How do the peculiarities mentioned above affect the continuity equation? To answer the question, we revise the corresponding conventional case [13]. According to it, the wave function satisfies the continuity equation

$$\frac{\partial|\Psi|^2}{\partial t} + \operatorname{div}\mathbf{j} = 0, \tag{9.1}$$

where $\mathbf{j} \equiv (i\hbar/2m)(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)$ is the probability current density.

Equation (9.1) is a direct consequence of Schrödinger's equation and the identities

$$\hat{H} \equiv \hat{H}^*; \quad \Psi\Delta\Psi^* - \Psi^*\Delta\Psi \equiv \operatorname{div}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi). \tag{9.2}$$

Conditions equation (9.2) hold for the problem in question, and hence, equation (9.1) is valid in this case too. However, now, due to the time dependence of the norm and (ii), neither $|\Psi|^2$ is the probability density, nor $(i\hbar/2m)(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)$ is the probability density current. The corresponding quantities are $\rho = |\Psi|^2/||\Psi||^2$ and $\mathbf{J} = (i\hbar/2m||\Psi||^2)(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)$, respectively.

Then, instead of equation (9.1) the continuity equation reads

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{J} = -\frac{\rho}{||\Psi||^2} \frac{\mathrm{d}||\Psi||^2}{\mathrm{d}t} = -\left(\frac{3}{2} + 2\mu'\right) \frac{\rho}{t}. \tag{9.3}$$

To write the last equality in equation (9.3) we have employed the expression $||\Psi||^2 = C(-\chi t)^{(3/2)+2\mu'}$.

If we integrate equation (9.3) over a certain closed volume V, the integral of $\partial\rho/\partial t$ gives the rate of the probability variation of finding the particle in this volume. The integral of $\operatorname{div} \mathbf{J}$ is transformed into the integral over the bounding V surface(s): it is the flux of the probability density current through this surface(s).

In the conventional case, the r.h.s. of the continuity equation is zero [13], and the flux is the only cause of the probability variations. In contrast, since equation (9.3) has a non-zero r.h.s., the latter also contributes to the probability variations. Notably, the meaning of this contribution is the generation or leakage (depending on the sign of the r.h.s.) of the probability density inside V.

Seemingly, this feature corresponds to creating something from nothing (generation) or vice versa (leakage). However, it does not! Such a behaviour directly follows from the variations of the *size* of the wave function spatial localization region due to the collapse (escape). Indeed, let us consider a particle in a square potential well with infinitely high walls, as an example. Suppose, these walls "adiabatically" move either toward or opposite each other. Then, obviously, the probability of finding the particle in a given part of the space inside the well changes in time, while the probability flux through the walls is zero: the changes are caused solely by the variations of the size of the wave function localization region. The same effect takes place during the collapse (escape).

Now, we integrate equation (9.3) over all space. Since, by definition, in this case, $\int \rho d^3\mathbf{r} \to 1$, equation (9.3) yields

$$\int \operatorname{div} \mathbf{J} d^3\mathbf{r} = -\left(\frac{3}{2} + 2\mu'\right) \frac{1}{t}. \tag{9.4}$$

Regarding the integral on the l.h.s. of equation (9.4), placing two concentric spheres about the point $r = 0$ and employing Gauss's theorem, we reduce the integral over the volume to the ones over the surfaces of the spheres. Then, we let the inner and outer spheres' radii tend to zero and infinity, respectively. Calculations based on the asymptotical expressions for $R(\xi)$ at $\xi \to 0$ and $\xi \to \infty$ presented in appendix B show that, in this case, the flux through the outer sphere tends to zero, while for the inner sphere, this is not the case, namely the singularity at $r = 0$ acts as a sink, at $\mu' > -3/4$, and as a source, at $\mu' < -3/4$, in entire agreement with the expression $||\Psi||^2 = C(-\chi t)^{(3/2)+2\mu'}$.

It is important to stress that existence of the sink (source) at $r = 0$ does not mean the particle gradually gets out (in) our world to (from) the singularity. Due to the selected normalization rule (ii), the probability of finding the particle in all space remains fixed and always equals unity.

10. Specific examples

To illustrate the behaviour of the obtained solutions, we consider specific examples of equation (6.6) at $\mu = 0$,

$$C_1 = \left(\frac{i}{2}\right)^{-(i\alpha/4)} \Gamma\left(1 + \frac{i\alpha}{2}\right) \Gamma\left(\frac{5 - i\alpha}{4}\right) \tag{10.1}$$

and

$$C_2 = -\left(\frac{i}{2}\right)^{i\alpha/4} \Gamma\left(1 - \frac{i\alpha}{2}\right) \Gamma\left(\frac{5 + i\alpha}{4}\right), \tag{10.2}$$

and several characteristic values of α. Here $\Gamma(x)$ stands for the Euler gamma function. At $\mu = 0$, the choice of $C_{1,2}$ in the form of equations (10.1), (10.2) ensures the convergence of the norm integral; see appendix B, equation (B 11).

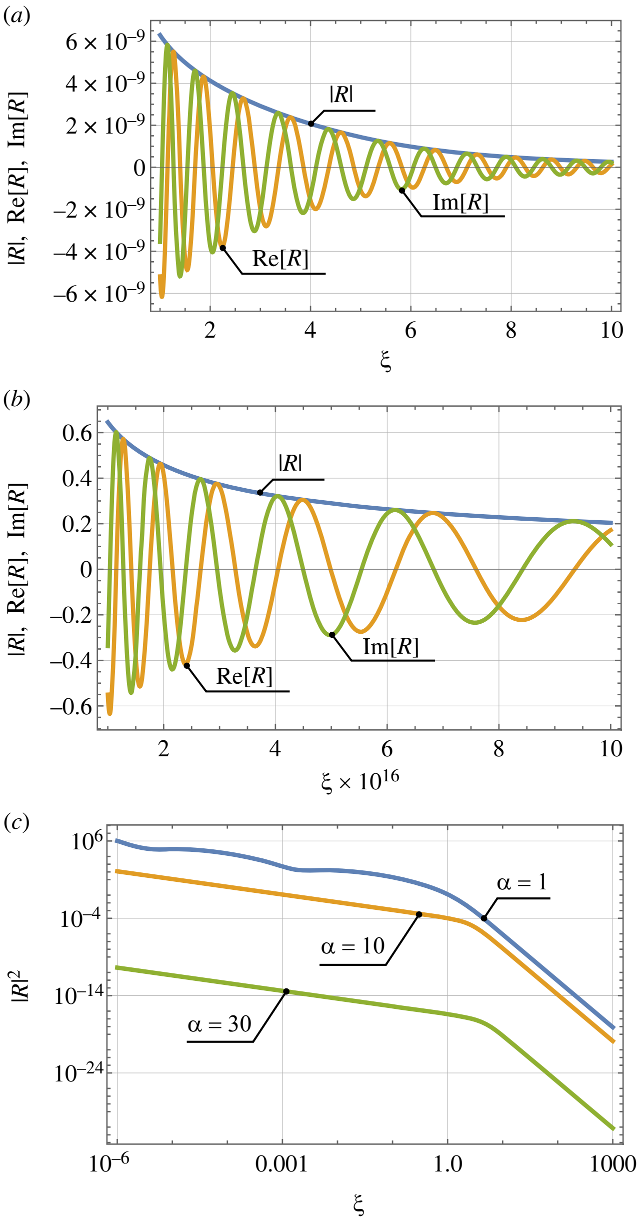

Figure 2. (*a*) and (*b*) Modulus, real and imaginary parts of $R(\xi)$ given by equations (6.6), in the specific case of equations (10.1), (10.2), $\mu = 0$, and $\alpha = 30$; Note the *16 orders of magnitude difference* in the characteristic scales of ξ in (*a*) and (*b*). (*c*) Log-log plots of $|R(\xi)|^2$ in the case of equations (10.1), (10.2), and $\mu = 0$, at $\alpha = 1, 10, 30$. Two power-law asymptotics: $|R|^2 \sim 1/\xi$, at $\xi \to 0$; and $|R|^2 \sim \xi^{-6}$, at $\xi \to \infty$ (see appendix), are well-pronounced.

The corresponding plots are shown in figure 2. Though the ξ-scales in figures 2*a*,*b* differ in *16 orders of magnitude*, $\mathrm{Re}\, R(\xi)$ and $\mathrm{Im}\, R(\xi)$ keep the same self-similar profiles, whose characteristic scale monotonically tends to zero at $\xi \to 0$. In particular, the phase shift in oscillations of $\mathrm{Re}\, R(\xi)$ and $\mathrm{Im}\, R(\xi)$ remains fixed at any ξ so that the zeros of one function correspond to the local extrema of the other and vice versa. As a result, the oscillations of the real and imaginary parts of $R(\xi)$ do not affect its modulus, which is a smooth monotonic function of ξ. This feature is generic and does not depend on the value of α, at least if α is not too small, see figure 2*c*. Notably, such a peculiarity has nothing to do with the self-similarity of equations (6.1), (6.4). The latter affects only the specific choice of ξ, not the behaviour of $R(\xi)$ at $\xi \to 0$.

11. Conclusion

Thus, we have studied and compared non-regularized classical and quantum collapses (escapes). We constructively have proven that close to the completion of the collapse (the beginning of the escape), the general solution to the classical problem is transformed into the self-similar one, i.e. the latter is an attractor for any other solutions. We also have shown that the classical collapse is continuously transformed into the escape and vice versa, i.e. the end of either marks the beginning of the other. Calculating the radiative losses for a specific example of the collapse corresponding to the fall of an electron with zero angular momentum to a neutral atom (molecule) with a finite dipole moment, we have shown that long before the impact of the radiative losses on the collapse dynamics becomes noticeable, the classical description must be replaced by the quantum one.

Then, we have proven the existence of quantum collapse and escape by obtaining the family of exact solutions to Schrödinger's equation, describing the phenomena. By simple arguments of dimensional analysis supplemented by the matching of the general quantum and classical cases through the quasi-classical approximation, we have shown that the obtained self-similar solutions to the quantum problem also are attractors for a much broader class of non-self-similar ones.

Since, for the obtained exact solutions to Schrödinger's equation, the norm of the wave function is time-depended, we have generalized to this case the conventional rule to calculate the mean values of operators, derived the corresponding continuity equation, and discussed its properties. We also have revealed a striking similarity between the classical and quantum collapses. Presumably, this fact should be related to certain hidden symmetry of the problem, insensitive to its classical or quantum nature.

Note that two- and one-dimensional versions of quantum collapse are also meaningful [23,25,30,33,40,44]. Since the spatial dimension does not affect the scaling properties of Schrödinger's equation required for self-similarity, the approach developed in the present paper may be straightforwardly applied to these problems too. In these cases, the self-similar variable ξ remains the same, while the governing equation (6.4) becomes different. Detailed discussions of these issues lie beyond the scope of our analysis.

Thus, the self-similar solutions introduced here are a convenient and powerful tool to investigate various dynamical effects in classical and quantum mechanics. Hopefully, the presented study sheds new light on the fundamentals of quantum mechanics and provides a better understanding of its basic principles.

Data accessibility. This article has no additional data.

Authors' contributions. M.I.T.: conceptualization, formal analysis, funding acquisition, investigation, methodology, writing—original draft, writing—review and editing.

Conflict of interest declaration. I declare I have no competing interests.

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Appendix A. Self-Similar version of Schrödinger's equation

Here, we find the conditions for the reduction of Schrödinger's equation to a self-similar form. One of the most general ansatzes for the wave function with a given ℓ is as follows:

$$\Psi = \Phi(-\chi t)R(\xi)Y_\ell^m(\theta, \varphi), \tag{A 1}$$

where $\xi = r/(-\chi t)^{\nu}$, while Φ and R are yet unknown functions. For this type of $\Psi(r,t)$, we have $\partial^n\Psi/\partial r^n = (-\chi t)^{-n\nu}\partial^n\Psi/\partial\xi^n$. Regarding $\partial\Psi/\partial t$, it is given by the following expression:

$$\frac{\partial\Psi}{\partial t} = \frac{\nu\xi\chi\Phi(z)R'(\xi)}{z} - \chi R(\xi)\Phi'(z). \tag{A 2}$$

Here $z \equiv -\chi t$. Our goal is to reduce Schrödinger's equation to an equation depending solely on ξ and independent of t explicitly. The necessary condition for that is that $\Phi(z)/z$ and $\Phi'(z)$ on the l.h.s of Schrödinger's equation have the same dependence on t so that it makes a common for them factor, which may be cancelled with the same factor on the r.h.s. It means that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}z} = \mu\frac{\Phi}{z}, \tag{A 3}$$

where μ is an arbitrary constant. Integrating this equation, we readily obtain

$$\Phi = \mathrm{const}\cdot z^{\mu}. \tag{A 4}$$

Substituting equation (A 1) with this form of $\Phi(z)$ into equation (4.1) with $U(r)$ given by equation (4.2) we arrive at the following equation:

$$z^{1-2\nu}R''(\xi) + 2\left[\frac{im\nu\xi\chi}{\hbar} + \frac{z^{1-2\nu}}{\xi}\right]R'(\xi) + \left[\frac{2\beta m z^{1-\nu s}}{\hbar^2\xi^s} - \frac{2i\mu m\chi}{\hbar} - \frac{\ell(\ell+1)z^{1-2\nu}}{\xi^2}\right]R(\xi) = 0. \tag{A 5}$$

It is seen straightforwardly that equation (A 5) does not depend on z if and only if $\nu = 1/2$ and $s = 1/\nu = 2$. Regarding χ, the choice $\chi = \hbar/m$ is just a matter of convenience to turn the corresponding coefficient to unity. Then, equation (A 5) is transformed into equation (6.4).

Appendix B. Norm convergence

Here, we discuss the convergence of $\int_0^\infty |R(\xi)|^2\xi^2\,\mathrm{d}\xi$, where $R(\xi)$ is given by equation (6.6). Since ${}_1F_1(a;b;z)$ is an analytic function of z on the whole complex plane [50] only the convergence at the lower and upper limits should be examined.

The lower limit case is simple. Taking into account that ${}_1F_1(a;b;z) = 1 + O(z)$ at $z \to 0$ [50], we readily obtain that in proximity of $\xi = 0$ the most singular terms in the solution give rise to the expression

$$|R(\xi)| \approx \frac{1}{\sqrt{\xi}}|C_1\xi^{-(i\alpha/2)} + C_2\xi^{i\alpha/2}| \le \frac{|C_1| + |C_2|}{\sqrt{\xi}}. \tag{B 1}$$

This means, the singularity of $|R(\xi)|^2\xi^2$ at $\xi = 0$ is integrable. Remarkably, estimate (B 1) does not depend on μ. Thus, the integral $\int_0^\infty |R(\xi)|^2\xi^2\,\mathrm{d}\xi$ converges at the lower limit at any μ.

The case $\xi \to \infty$ is more tricky. The asymptotic expansion of ${}_1F_1(a;b;z)$ at $|z| \to \infty$ reads [50]

$$\begin{aligned} {}_1F_1(a;b;z) = {} & \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}G(a, a-b+1, -z) \\ & + \frac{\Gamma(b)}{\Gamma(a)}e^z z^{a-b}G(b-a, 1-a, z), \end{aligned} \tag{B 2}$$

where

$$G(a,b,z) = 1 + \frac{ab}{1!z} + \frac{a(a+1)b(b+1)}{2!z^2} + \cdots = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!z^n}. \tag{B 3}$$

In equation (B 3) $(x)_n$ designates the rising factorial (the Pochhammer function) defined as follows:

$$\begin{aligned} (x)_0 &= 1, \\ (x)_n &= x(x+1)(x+2)\cdots(x+n-1) \\ &= \prod_{k=0}^{n-1}(x+k) = \frac{\Gamma(x+n)}{\Gamma(x)}. \end{aligned} \tag{B 4}$$

Then, according to equations (B 2), (B 3), the asymptotic expansion for ${}_1F_1(a;b;z)$ has two groups of terms originated in the expressions proportional to $(-z)^{-a}G$ and $e^z\, z^{a-b}G$, respectively. According to equation (B 3), to select the most singular terms in every group at $z\to\infty$, we have to replace G by 1. Then, equation (6.6) results in the following asymptotic expression for $R(\xi)\approx {}_1R^{(0)}(\xi)+{}_2R^{(0)}(\xi)$, where

$$ {}_1R^{(0)}(\xi)={}_1C^{(0)}\xi^{2\mu}\times\left[\frac{(i/2)^{i\alpha/4}\Gamma(1-(i\alpha/2))}{\Gamma(((5-i\alpha)/4)+\mu)}C_1+\frac{(i/2)^{-(i\alpha/4)}\Gamma(1+(i\alpha/2))}{\Gamma(((5+i\alpha)/4)+\mu)}C_2\right] \tag{B 5}$$

and

$$ {}_2R^{(0)}(\xi)={}_2C^{(0)}\,e^{-(i\xi^2/2)}\xi^{-3-2\mu} \times\left[\frac{(-(i/2))^{i\alpha/4}\Gamma(1-(i\alpha/2))}{\Gamma(-((1+i\alpha)/4)-\mu)}C_1+\frac{(-(i/2))^{-(i\alpha/4)}\Gamma(1+(i\alpha/2))}{\Gamma(-((1-i\alpha)/4)-\mu)}C_2\right], \tag{B 6}$$

Here ${}_{1,2}C^{(0)}$ are constants. The cumbersome explicit expressions for them are obtained upon the step-by-step implementation of the above procedure. We do not need these expressions for further analysis.

With the proper choice of the ratio C_1/C_2 we turn to zero either ${}_1R^{(0)}(\xi)$ or ${}_2R^{(0)}(\xi)$. If instead of the leading terms solely, we employ the entire infinite series equation (B 3), the expression for $R(\xi)$ becomes the following:

$$ R(\xi)=\sum_{n=0}^{\infty}[{}_1R^{(n)}(\xi)+{}_2R^{(n)}(\xi)]. \tag{B 7}$$

A remarkable thing, however, is that the expressions for ${}_{1,2}R^{(n)}(\xi)$ preserve the same structure as that in equations (B 5), (B 6). The only difference is in the change of the prefactor: ${}_1C^{(0)}\xi^{2\mu}\to {}_1C^{(n)}\xi^{2(\mu-n)}$ for ${}_1R^{(n)}(\xi)$ and ${}_2C^{(0)}\,e^{-(i\xi^2/2)}\xi^{-3-2\mu}\to {}_2C^{(n)}\,e^{-(i\xi^2/2)}\xi^{-3-2(\mu+n)}$ for ${}_2R^{(n)}(\xi)$; the expressions in the square delimiters in equations (B 5), (B 6) do not depend on n and hence remain the same at any n. It means, that the value of the ratio C_2/C_1 which turns ${}_jR^{(0)}$ to zero (here $j=1,2$) simultaneously turns to zero all ${}_jR^{(n)}$ with the same value of j and any value of n, i.e. the entire infinite series $\sum_{n=0}^{\infty}{}_jR^{(n)}(\xi)$ vanishes.

Thus, at

$$ \frac{C_2}{C_1}=-\frac{(-(i/2))^{i\alpha/2}\Gamma(1-(i\alpha/2))\Gamma(-((1-i\alpha)/4)-\mu)}{\Gamma(1+(i\alpha/2))\Gamma(-((1+i\alpha)/4)-\mu)}, \tag{B 8}$$

${}_2R^{(n)}=0$ at any n, and $R(\xi)=\sum_{n=0}^{\infty}{}_1R^{(n)}(\xi)$.

At

$$ \frac{C_2}{C_1}=-\frac{(i/2)^{i\alpha/2}\Gamma(1-(i\alpha/2))\Gamma(((5+i\alpha)/4)+\mu)}{\Gamma(1+(i\alpha/2))\Gamma(((5-i\alpha)/4)+\mu)}, \tag{B 9}$$

all ${}_1R^{(n)}$ vanish, and $R(\xi)=\sum_{n=0}^{\infty}{}_2R^{(n)}(\xi)$.

In both cases, the most singular term is the one with $n=0$. Let us inspect these terms' contribution to the norm's integral for the obtained wave function. In case equation (B 8), $|R(\xi)|^2\sim\xi^{4\mu'}$. Then, at $\xi\to\infty$, the integral $\int|R(\xi)|^2\xi^2\,d\xi\sim\xi^{4\mu'+3}$. Its convergence requires $\mu'<-3/4$.

Similarly, in case equation (B 9), $|R(\xi)|^2\sim\xi^{-6-4\mu'}$. Then, the norm integral converges at the upper limit, provided $\mu'>-3/4$.

At $\mu'=-3/4$, the integral $\int_0^{\infty}|R(\xi)|^2\xi^2\,d\xi$ diverges at the upper limit as $\ln\xi$ owing to the contribution of $|{}_{1,2}R^{(0)}(\xi)|^2$. Since the sole value of μ' when the norm of the obtained wave function does not depend on time is $\mu'=-3/4$, only time-depended norms are physically meaningful for the given wave function.

It is convenient to present the explicit form of the obtained solutions admitting the normalization. It is as follows:

At $\mu' < -3/4$

$$\begin{aligned} R(\xi) = \frac{C}{\sqrt{\xi}} \Bigg[& \left(-\frac{i\xi^2}{2}\right)^{-(i\alpha/4)} \Gamma\left(1+\frac{i\alpha}{2}\right) \Gamma\left(-\frac{1+i\alpha}{4}-\mu\right) \\ & \times {}_1F_1\left(-\frac{1+i\alpha}{4}-\mu; 1-\frac{i\alpha}{2}; -\frac{i\xi^2}{2}\right) \\ & - \left(-\frac{i\xi^2}{2}\right)^{i\alpha/4} \Gamma\left(1-\frac{i\alpha}{2}\right) \Gamma\left(-\frac{1-i\alpha}{4}-\mu\right) \\ & \times {}_1F_1\left(-\frac{1-i\alpha}{4}-\mu; 1+\frac{i\alpha}{2}; -\frac{i\xi^2}{2}\right) \Bigg]. \end{aligned} \tag{B 10}$$

At $\mu' > -3/4$

$$\begin{aligned} R(\xi) = \frac{C}{\sqrt{\xi}} \Bigg[& \left(\frac{i\xi^2}{2}\right)^{-(i\alpha/4)} \Gamma\left(1+\frac{i\alpha}{2}\right) \Gamma\left(\frac{5-i\alpha}{4}+\mu\right) \\ & \times {}_1F_1\left(-\frac{1+i\alpha}{4}-\mu; 1-\frac{i\alpha}{2}; -\frac{i\xi^2}{2}\right) \\ & - \left(\frac{i\xi^2}{2}\right)^{i\alpha/4} \Gamma\left(1-\frac{i\alpha}{2}\right) \Gamma\left(\frac{5+i\alpha}{4}+\mu\right) \\ & \times {}_1F_1\left(-\frac{1-i\alpha}{4}-\mu; 1+\frac{i\alpha}{2}; -\frac{i\xi^2}{2}\right) \Bigg]. \end{aligned} \tag{B 11}$$

For the sake of symmetry, we have rescaled the constant of integration in equations (B 5), (B 6) so that for equation (B 10)

$$C_1 = \left(-\frac{i}{2}\right)^{-(i\alpha/4)} \Gamma\left(1+\frac{i\alpha}{2}\right) \Gamma\left(-\frac{1+i\alpha}{4}-\mu\right) C, \tag{B 12}$$

while for equation (B 11)

$$C_1 = \left(\frac{i}{2}\right)^{-(i\alpha/4)} \Gamma\left(1+\frac{i\alpha}{2}\right) \Gamma\left(\frac{5-i\alpha}{4}+\mu\right) C. \tag{B 13}$$

Note that the expressions in square delimiters on the r.h.s.' of both equations (B 10) and (B 11) are the differences of the two terms, where the second term is obtained from the first by the formal transformation $\alpha \to -\alpha$. At $\alpha = 0$ they are identical, and the r.h.s.' of equations (B 10), (B 11) vanish. Since the necessary collapse condition reads $\alpha \equiv \sqrt{4\gamma - 1} > 0$, see equation (6.7), it means that the obtained solutions smoothly vanish at the continuous transformation of the potential from the collapsing to the non-collapsing type at $\alpha \to 0$.

Appendix C. Uncertainty relations

For the reader's convenience, we reproduce here the main points of the derivation of the uncertainty relations presented in [13], to show that it indeed does not require the Hamiltonian self-adjointness. First of all, we recall that in the coordinate representation, the momentum operator reads

$$\hat{\mathbf{p}} = -i\hbar\frac{\partial}{\partial \mathbf{r}}, \tag{C 1}$$

Then, we suppose that a particle with the mean value of the momentum $\mathbf{p}_0$ is localized in a finite region of space with the sizes Δx, Δy and Δz. In this case, its wave function has the form $\psi = u(\mathbf{r})\exp[(i\mathbf{r}\mathbf{p}_0/\hbar)]$, where $u(\mathbf{r})$ sharply decays beyond the localization region.

Next, the eigenfunctions of the operator $\hat{\mathbf{p}}$ are plane waves so that the corresponding eigenfunction expansion of a wave function ψ is just a Fourier integral. The coefficients in this integral $\psi_{\mathbf{p}}$ are the Fourier transforms of ψ

$$\psi_{\mathbf{p}} = \int u(\mathbf{r})\exp[i\mathbf{r}(\mathbf{p}_0 - \mathbf{p})/\hbar]\, d^3\mathbf{r}. \tag{C 2}$$

Since the mentioned sharp decay of $u(\mathbf{r})$ beyond the localization region, this region makes an overwhelming contribution to the integral on the r.h.s. of equation (C 2). On the other hand, if we consider $\psi_{\mathbf{p}}$ as a function of $\Delta p_{x,y,z} = |p_{0x,y,z} - p_{x,y,z}|$, we will see its rapid decay as soon as $\Delta p_{x,y,z}$ exceed the values

$$\Delta p_x \Delta x \sim \hbar,\ \Delta p_y \Delta y \sim \hbar,\ \Delta p_z \Delta z \sim \hbar. \tag{C 3}$$

The decay is related to the rapid oscillations of $\exp[i\mathbf{r}(\mathbf{p}_0 - \mathbf{p})/\hbar]$ in the area of the main contribution to the integral, if $\Delta p_{x,y,z}$ occurs beyond the specified bounds.

To complete the derivation, we have to recall that the probability density to find a given value of $\mathbf{p}$ is proportional to $|\psi_{\mathbf{p}}|^2$. Thus, the probability for $\Delta p_{x,y,z}$ to have a value beyond the bounds equation (C 3) is negligibly small.

References

1. Guderley K. 1942 Starke kugelige und zylindrische verdichtungsstosse in der nahe des kugelmitterpunktes bnw. der zylinderachse. *Luftfahrtforschung* **19**, 302.
2. Stanyukovich KP. 2016 *Unsteady motion of continuous media*. Elsevier.
3. Zel'dovich YB, Raizer YP. 1967 *Physics of shock waves and high-temperature hydrodynamic phenomena*, Chap. XII, §2. New York, NY: Academic Press.
4. Lord R. 1917 On the pressure developed in a liquid during the collapse of a spherical cavity. *London, Edinb., Dublin Phil. Mag. J. Sci.* (*6th Series*) **34**, 94–98. (doi:10.1080/14786440808635681)
5. Hunter C. 1960 On the collapse of an empty cavity in water. *J. Fluid Mech.* **8**, 241–263. (doi:10.1017/S0022112060000578)
6. Zababakhin EI. 1960 The collapse of bubbles in a viscous liquid. *Appl. Math. Mech., PMM* (*English Transl.*) **24**, 1714–1717. (doi:10.1016/0021-8928(60)90025-3)
7. Askaryan G. 1962 Cerenkov radiation and transition radiation from electromagnetic waves. *Sov. Phys. JETP* **15**, 943.
8. Chiao RY, Garmire E, Townes CH. 1964 Self-trapping of optical beams. *Phys. Rev. Lett.* **13**, 479–482. (doi:10.1103/PhysRevLett.13.479)
9. Kelley PL. 1965 Self-focusing of optical beams. *Phys. Rev. Lett.* **15**, 1005–1008. (doi:10.1103/PhysRevLett.15.1005)
10. Arnol'd VI. 2013 *Mathematical methods of classical mechanics*, vol. 60. Springer Science & Business Media.
11. Zakharov VE. 1972 Collapse of Langmuir waves. *Sov. Phys. JETP* **35**, 908–914.
12. Zakharov VE, Kuznetsov EA. 2012 Solitons and collapses: two evolution scenarios of nonlinear wave systems. *Phys. Usp.* **55**, 535. (doi:10.3367/UFNe.0182.201206a.0569)
13. Landau LD, Lifshitz EM. 2013 *Quantum mechanics: non-relativistic theory*. Elsevier.
14. Morse P, Feshbach H. 1953 *Methods of theoretical physics*, vol. 2, §12.3. MGH.
15. Betan RI, Nazarewicz W. 2012 Alpha decay in the complex energy shell model. *Phys. Rev. C* **86**, 034338. (doi:10.1103/PhysRevC.86.034338)
16. Fano U. 1961 Effects of configuration interaction on intensities and phase shifts. *Phys. Rev.* **124**, 1866–1878. (doi:10.1103/PhysRev.124.1866)
17. Shortley GH. 1931 The inverse-cube central force field in quantum mechanics. *Phys. Rev.* **38**, 120. (doi:10.1103/PhysRev.38.120)

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18. Case K. 1950 Singular potentials. *Phys. Rev.* **80**, 797. (doi:10.1103/PhysRev.80.797)
19. Guggenheim E. 1966 The inverse square potential field. *Proc. Phys. Soc.* **89**, 491. (doi:10.1088/0370-1328/89/3/302)
20. Lévy-Leblond JM. 1967 Electron capture by polar molecules. *Phys. Rev.* **153**, 1. (doi:10.1103/PhysRev.153.1)
21. Efimov V. 1971 Weakly-bound states of three resonantly-interacting particles. *Sov. J. Nucl. Phys.* **12**, 589–553.
22. Alliluev S. 1972 The problem of collapse to the center in quantum mechanics. *Sov. Phys. JETP* **34**.
23. Parisi G, Zirilli F. 1973 Anomalous dimensions in one-dimensional quantum field theory. *J. Math. Phys.* **14**, 243–245. (doi:10.1063/1.1666302)
24. Schwartz C. 1976 Almost singular potentials. *J. Math. Phys.* **17**, 863–867. (doi:10.1063/1.522998)
25. Van Haeringen H. 1978 Bound States for r^{-2}-like Potentials in one and three dimensions. *J. Math. Phys.* **19**, 2171–2179. (doi:10.1063/1.523574)
26. Wu H, Sprung D. 1994 Inverse-square potential and the quantum vortex. *Phys. Rev. A* **49**, 4305. (doi:10.1103/PhysRevA.49.4305)
27. Song DY. 2000 Unitary relation between a harmonic oscillator of time-dependent frequency and a simple harmonic oscillator with and without an inverse-square potential. *Phys. Rev. A* **62**, 014103. (doi:10.1103/PhysRevA.62.014103)
28. Camblong HE, Epele LN, Fanchiotti H, García Canal CA. 2000 Renormalization of the inverse square potential. *Phys. Rev. Lett.* **85**, 1590–1593. (doi:10.1103/PhysRevLett.85.1590)
29. Dong SH, Lozada-Cassou M. 2005 An algebraic approach to the harmonic oscillator plus an inverse square potential in three dimensions. *Am. J. Appl. Sci.* **2**, 376–382. (doi:10.3844/ajassp.2005.376.382)
30. Ávila-Aoki M, Cisneros C, Martínez-y-Romero RP, Núñez-Yépez HN, Salas-Brito AL. 2009 Classical and quantum motion in an inverse square potential. *Phys. Lett. A* **373**, 418–421. (doi:10.1016/j.physleta.2008.11.054)
31. Astrakharchik GE, Malomed BA. 2015 Quantum versus mean-field collapse in a many-body system. *Phys. Rev. A* **92**, 043632. (doi:10.1103/PhysRevA.92.043632)
32. Malomed B. 2018 Suppression of quantum-mechanical collapse in bosonic gases with intrinsic repulsion: a brief review. *Condens. Matter* **3**, 15. (doi:10.3390/condmat3020015)
33. Sakaguchi H, Malomed BA. 2011 Suppression of quantum collapse in an anisotropic gas of dipolar bosons. *Phys. Rev. A* **84**, 033616. (doi:10.1103/PhysRevA.84.033616)
34. Sakaguchi H, Malomed BA. 2013 Suppression of the quantum collapse in binary bosonic gases. *Phys. Rev. A* **88**, 043638. (doi:10.1103/PhysRevA.88.043638)
35. Sakaguchi H, Malomed BA. 2011 Suppression of the quantum-mechanical collapse by repulsive interactions in a quantum gas. *Phys. Rev. A* **83**, 013607. (doi:10.1103/PhysRevA.83.013607)
36. Shamriz E, Chen Z, Malomed BA. 2020a Suppression of the quasi-two-dimensional quantum collapse in the attraction field by the Lee-Huang-Yang effect. *Phys. Rev. A* **101**, 063628. (doi:10.1103/PhysRevA.101.063628)
37. Shamriz E, Chen Z, Malomed BA, Sakaguchi H. 2020b Singular mean-field states: a brief review of recent results. *Condens. Matter* **5**, 20. (doi:10.3390/condmat5010020)
38. Frank WM, Land DJ, Spector RM. 1971 Singular potentials. *Rev. Mod. Phys.* **43**, 36. (doi:10.1103/RevModPhys.43.36)
39. Landau LD, Lifshitz EM. 1976 *Mechanics*, §14. Oxford, UK: Pergamon Press.
40. Camblong HE, Ordóñez CR. 2003 Anomaly in conformal quantum mechanics: from molecular physics to black holes. *Phys. Rev. D* **68**, 125013. (doi:10.1103/PhysRevD.68.125013)
41. Efimov V. 1973 Energy levels of three resonantly interacting particles. *Nucl. Phys. A* **210**, 157–188. (doi:10.1016/0375-9474(73)90510-1)
42. Denschlag J, Schmiedmayer J. 1997 Scattering a neutral atom from a charged wire. *Europhys. Lett.* **38**, 405. (doi:10.1209/epl/i1997-00259-y)
43. Kulkarni M, Abanov AG. 2011 Cold Fermi gas with inverse square interaction in a harmonic trap. *Nucl. Phys. B* **846**, 122–136. (doi:10.1016/j.nuclphysb.2010.12.015)
44. Nisoli C, Bishop A. 2014 Attractive inverse square potential, U (1) gauge, and winding transitions. *Phys. Rev. Lett.* **112**, 070401. (doi:10.1103/PhysRevLett.112.070401)
45. Claus P, Derix M, Kallosh R, Kumar J, Townsend PK, Van Proeyen A. 1998 Black holes and superconformal mechanics. *Phys. Rev. Lett.* **81**, 4553. (doi:10.1103/PhysRevLett.81.4553)

46. Barenblatt GI. 1996 *Scaling, self-similarity, and intermediate asymptotics: dimensional analysis and intermediate asymptotics*. Cambridge, UK: Cambridge University Press.
47. Sedov LI. 1993 *Similarity and dimensional methods in mechanics*. Boca Raton, FL: CRC Press.
48. Landau LD. 2013 *The classical theory of fields*, §67, vol. 2. Elsevier.
49. Zakharov V, Kuznetsov E. 1986 Quasi-classical theory of threedimensional wave collapse. *Zh. Eksp. Teor. Fiz* **91**, 1310.
50. Janke E, Emde F, Lösch F. 1960 *Tafeln höherer funktionen*. Stuttgart: BG Teubner.