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Three-leg form factor on Coulomb branch

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Abstract: We study the form factor of the lowest component of the stress-tensor multiplet away from the origin of the moduli space in the spontaneously broken, aka Coulomb, phase of the maximally supersymmetric Yang-Mills theory for decay into three massive W-bosons. The calculations are done at two-loop order by deriving and solving canonical differential equations in the asymptotical limit of nearly vanishing W-masses. We confirm our previous findings that infrared physics of 'off-shell observables' is governed by the octagon anomalous dimension rather than the cusp. In addition, the form factor in question possesses a nontrivial remainder function, which was found to be identical to the massless case, upon a proper subtraction of infrared logarithms (and finite terms). However, the iterative structure of the object is more intricate and is not simply related to the previous orders in coupling as opposed to amplitudes/form factors at the origin of the moduli space.

Keywords: Extended Supersymmetry, Higher-Order Perturbative Calculations, Scattering Amplitudes, Supersymmetric Gauge Theory

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1 Introduction

The so-called Coulomb branch of the spontaneously broken maximally supersymmetric Yang-Mills (sYM) theory [1] is a natural laboratory to study off-shell amplitudes and form factors in four-dimensional gauge theories. Endowing some (or all) scalars of the model with vacuum expectation values, one can adjust their values in a way as to yield matrix elements which possess massive external states, i.e., W-bosons, Higgs-like scalars etc., but only massless excitations propagating in quantum loops.

The $\mathcal{N} = 4$ model away from the origin of the moduli space can naturally be obtained from a generalized form of the dimensional reduction [2–4] akin to the original one used to discover its Lagrangian in the first place from the ten-dimensional $\mathcal{N} = 1$ sYM [5, 6], or the six-dimensional $\mathcal{N} = (1, 1)$ sYM (see, e.g., [7]). Instead of setting the extra-dimensional, i.e., $D > 4$, components of momenta to zero, one can trade them in lieu of the scalars' moduli, i.e., vacuum averages of $D > 4$ components of the ten-dimensional gauge field. The 16 supercharges remain unbroken in this phase, but the supersymmetric algebra gets a central extension with BPS charges induced by nonvanishing masses, and thus the theory shares a gamut of properties of its conformal sibling. One-loop analyses demonstrated that the Coulomb branch scattering amplitudes obey a no-triangle rule,[1] thus enjoying only boxes in their integral expansion [3, 8] at a generic point of the moduli space, — a generalization of the background-field gauge proof from refs. [9, 10] applicable to off-shell massless amplitudes. Further, there are no rational terms as well [3, 11] and, therefore, integrands on the Coulomb branch are cut-constructible [12]. Making use of this latter property, a proof of the dual conformal invariance

[1]Bubbles and tadpoles are excluded form the get-go based on their poor ultraviolet properties.

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of massive loop integrals from a six-dimensional viewpoint was elucidated in refs. [7, 13]. Also correctness of the four-leg amplitudes at four-loop order (including nonplanar contributions) [7] was demonstrated to expressions built using solely four-dimensional momenta in the cuts [14] by lifting four-dimensional inner products of momenta up to six dimensions.

The above higher-dimensional perspective provides a natural bridge between the dimensionally regularized theory and its massive version to tame infrared divergences in scattering amplitudes and form factors in a gauge invariant manner. Their explicit structure for the four-gluon amplitude and the Sudakov form factor was deduced at up to three-loop level [1, 15–17] by promoting massless integral bases constructed in four dimensions to involve massive propagators only around graphs' periphery. Infrared structure was shown to be in accord with the well-known conformal phase of $\mathcal{N}=4$ sYM in $D=4-2\varepsilon$ [18] (see refs. [19–21], for earlier QCD studies) for a minor difference in kinematically-independent contributions and in compliance with a common wisdom that infrared properties of gauge theories are encoded in the so-called cusp anomalous dimension [22, 23].

The situation drastically changes, however, when all internal propagators are left massless, but the external legs are kept massive, or off-shell, as we will refer to them hereafter. Four- [24] and five-leg [25] W-boson amplitudes as well as the two-W-boson Sudakov form factor [26, 27] enjoyed the same recurrent feature in variance to the naive expectation: the infrared logarithms are governed by an exponent different from the cusp anomalous dimension. Instead they exhibit dependence on the so-called octagon anomalous dimension which made its debut in completely different circumstances: the light-cone limit of correlation functions of infinitely heavy BPS operators [28, 29] and the near-origin asymptotics of the six-gluon remainder function [30].

In the current paper, we continue our exploration of the Coulomb branch by addressing a more involved quantity, a three-particle form factor $\mathcal{F}_3$ of the chiral part of the stress tensor supermultiplet $\mathcal{T}$ [31]

$$\mathcal{T}(x,\theta_+) = \operatorname{tr}\phi_{++}^2(x) + \cdots + \theta_+^4\mathcal{L}(x)\,. \tag{1.1}$$

Here we displayed its lowest and highest components only. The lowest one is built from harmonic projections [32] of the sextet of the $\mathcal{N}=4$ scalars, while

$$\mathcal{L} = -\frac{1}{8}\operatorname{tr}(F_{\alpha\beta}+i\widetilde{F}_{\alpha\beta})^2 + \text{fermions and scalars}\,, \tag{1.2}$$

is the chiral on-shell Lagrangian. By considering a super-matrix element of $\mathcal{T}$ in the states created by the Nair super-wave function [33] $\Phi_i=\Phi(p_i,\theta_i)$, i.e., $\langle\Phi_1\Phi_2\Phi_3|\mathcal{T}(x,\theta_+)|0\rangle$, various terms in its θ-expansion are related by the Poincare supersymmetry generator Q, which obeys an algebra that closes off the mass shell [31]. In this manner, dependence on a particular choice of external states and/or operators involved enters solely through an overall kinematical tree-level factor, with all dynamical information encoded in a universal function of the coupling constant and Mandelstam-like invariants. In particular, a two-scalar plus gluon form factor of the lowest component of the chiral stress tensor multiplet considered in ref. [34] is the same as the three-gauge-boson form factor of $\mathcal{L}$, see section 3.2 of ref. [35]. So in the following, we will refer to it as the form factor of three W-bosons.

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Thus, the main observable of our consideration is

$$\int d^4x \, \mathrm{e}^{-iq\cdot x} \langle p_1, p_2, p_3 | \operatorname{tr} \phi_{12}^2(x) | 0 \rangle = (2\pi)^4 \delta^{(4)}(q - p_1 - p_2 - p_3) \mathcal{F}_3 \, . \tag{1.3}$$

Here we explicitly extracted the energy-momentum conserving delta function. This is the simplest 'observable' which possesses nontrivial remainder function after factoring out infrared divergences [34]. In the conformal phase of the theory, it was bootstrapped to a staggering eight-loop order [36, 37] using techniques adopted from scattering amplitudes [38]. Our goal will be much more modest: we will calculate its off-shell version at two loops. The incentive for our analysis is multifold. First, we would like to confirm the octagon anomalous dimension as the Sudakov exponent of 'off-shell observables'. Second, we will establish similarities/differences to the iterative structure of the form factor with increased perturbative order compared to its conformal analogue. Third, given that the infrared logarithms are different in the on- and off-shell cases, will the remainder functions differ as well?

Our subsequent presentation will be organized as follows. In the next section, we set up our notations. Then, in section 3, we perform the one-loop calculation, which is then followed by two loops in section 4. The only graph that was not touched upon in the existing literature corresponds to the tri-pentagon. So we perform its calculation from scratch in section 4.1. It is then followed by all other contributing graphs. In section 5, we add them up and use symbol analysis to simplify the sum and uncover the structure of the form factor at two-loop order. Finally, we conclude.

2 Setting up conventions

The form factor of three W-bosons contains an overall prefactor encoding polarization dependence of the external states. We will not be interested in it in what follows and thus introduce the ratio function

$$F_3 \equiv \mathcal{F}_3 / \mathcal{F}_{3,\mathrm{tree}} \, . \tag{2.1}$$

F_3 depends on three invariants s_{ij} and the off-shellnesses of the W-legs, which will be taken to have the same value μ,

$$s_{ij} \equiv (p_i + p_j)^2 \, , \quad p_i^2 \equiv -\mu \, . \tag{2.2}$$

These are linearly dependent, however,

$$s_{12} + s_{23} + s_{31} = q^2 - 3\mu \, . \tag{2.3}$$

Since the form factor is a homogeneous function of these kinematical variables, one can set one scale to one, e.g., $q^2 = -1$ below. Equivalently this can be done by introducing Mandelstam-like variables and the 'mass parameter' m

$$u = s_{12}/q^2 \, , \quad v = s_{23}/q^2 \, , \quad w = s_{31}/q^2 \, , \quad m = -\mu/q^2 \, , \tag{2.4}$$

and ignore the overall mass scale q^2: F_3 is dimensionless.

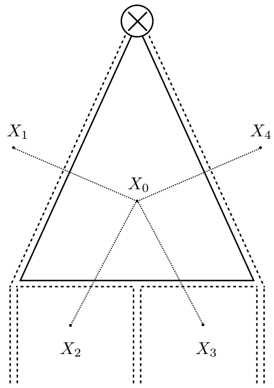

Figure 1. Double-line representation of a typical one-loop graph: the solid/dashed lines correspond to $\mathrm{SU}(N-M)$ and $\mathrm{SU}(M)$ groups, respectively. The dual graph is shown here as well with dotted lines. The external higher-dimensional coordinates X_i^M define the massless momenta $P_i^M = (X_i - X_{i+1})^M$ flowing through external legs. Their mass-shell condition $P_i^2 = 0$ provides non-vanishing masses/off-shellness in four dimensions, i.e., $(x_i - x_{i+1})^2 = (y_i - y_{i+1})^2$. On the other hand, the coordinate of the internal vertex X_0 is purely four-dimensional $X_0 = (x_0, 0)$.

Before we proceed with the required calculations, let us address the difference between the internal and external lines in Feynman graphs that determine our observable. This can be understood either from a purely four-dimensional point of view or from the vantage point of higher dimensions, either six or ten, as we already pointed out in the Introduction. The four-dimensional perspective was used in refs. [1] by introducing a Higgs mechanism into the $\mathcal{N}=4$ sYM and breaking the $\mathrm{SU}(N)$ gauge symmetry down to $\mathrm{SU}(N-M)\times\mathrm{SU}(M)$ with $N \gg M$. Then, providing expectation values to some of the scalars of the $\mathrm{SU}(M)$ subgroup yields masses for gauge bosons, aka W-bosons, and other excitations. A typical one-loop graph contributing to the form factor with the double-line color flow shown in figure 1 demonstrates that the heaviest states reside only outside the diagram with all inner lines being light/massless. The higher-dimensional approach [2, 3] provides a complementary understanding of mass generation. One can compactify the six-dimensional $\mathcal{N}=(1,1)$ or the ten-dimensional $\mathcal{N}=1$ sYM down to four but keep the out-of-four-dimensional components of fields momenta nonvanishing. The latter are then treated as complex masses. The dual graph representation in figure 1 then exhibits the necessary conditions one has to impose on the dual coordinates $X^M = (x^\mu, y^a)$ to induce nontrivial masses for external states while keeping the ones propagating in quantum loops massless, namely $(y_i - y_{i+1})^2 \neq 0$ and $y_i^2 = 0$, respectively.

F_3 admits perturbative series expansion in the gauge coupling g_{YM}^2 accompanied at each order by the number of colors N (in the planar limit), allowing us to introduce

$$g^2 \equiv \frac{g_{\mathrm{YM}}^2 N}{(4\pi)^2}(4\pi \mathrm{e}^{-\gamma_{\mathrm{E}}})^\varepsilon \,, \tag{2.5}$$

which comes hand-in-hand with a measure of the dimensionally-regularized loop momentum integrals,

$$(\mu_{\mathrm{DR}}^2 \mathrm{e}^{\gamma_{\mathrm{E}}})^\varepsilon \int \frac{d^D\ell}{i\pi^{D/2}} \,, \tag{2.6}$$

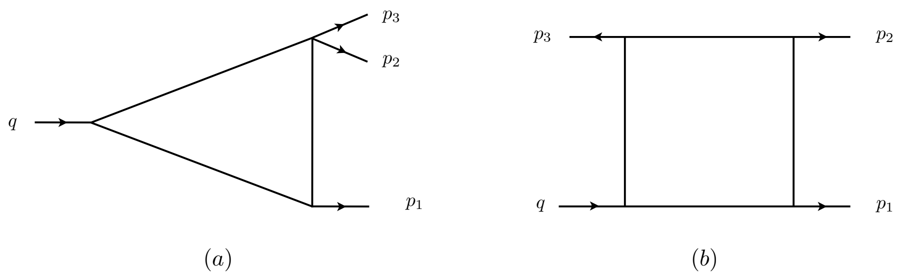

Figure 2. Graphs contributing to the one-loop form factor $F_3^{(1)}$.

in $D = 4 - 2\varepsilon$. We will dwell on the necessity to deal with the space-time away from $D = 4$, even though we already have an infrared regulator m, when it becomes indispensable at two-loop order.

Thus, we have to the lowest two orders

$$F_3 = 1 + g^2 F_3^{(1)} + g^4 F_3^{(2)} + \dots , \tag{2.7}$$

where $F_3^{(i)}$ are given by linear combinations of one- and two-loop integrals for $i = 1, 2$, respectively. Instead of using Feynman rules of the Coulomb phase of $\mathcal{N} = 4$ sYM in order to find the latter, we will employ, as was advocated in the introduction, the connection between the spontaneously broken phase in $D = 4$ and higher-dimensional theory with exact gauge symmetry to recycle generalized unitarity analyses from refs. [39, 40] and [34], to ascertain integral families defining the one and two-loop integrands, respectively. All calculations will be done in the limit $m \to 0$, i.e., they will be valid up to power corrections in m.

3 One loop

Without further ado, let us start our analysis with the one-loop form factor. At this order, $F_3^{(1)}$ receives the following expansion in terms of the triangle Tri and box Box integrals [39, 40]

$$F_3^{(1)} = \sum_{n=0}^{2} \mathbb{P}^n \left[(s_{12} + s_{13}) \text{Tri}(p_1, p_2 + p_3) + \frac{1}{2} s_{12} s_{23} \text{Box}(p_1, p_2, p_3) \right] , \tag{3.1}$$

shown in figure 2. In this equation, we introduced an operator $\mathbb{P}$ that shifts momentum indices of any function to its right by one

$$\mathbb{P} f_{ij\dots} = f_{i+1(\text{mod}\,3), j+1(\text{mod}\,3)\dots} , \tag{3.2}$$

modulo 3, which imposes periodicity. The Mandelstam-like variables then transform as $\mathbb{P}(u, v, w) = (v, w, u)$. Both integrals in eq. (3.1) can immediately be expressed in terms of the Davydychev-Ussyukina function $\Phi_1(x, y)$ [41, 42],

$$\Phi_\ell(x, y) = - \sum_{j=\ell}^{2\ell} \frac{j!(-1)^j \log^{2\ell-j}\left(\frac{y}{x}\right)}{\ell!(j-\ell)!(2\ell-j)!} \frac{\text{Li}_j\left(-(\rho x)^{-1}\right) - \text{Li}_j\left(-(\rho y)^{+1}\right)}{\lambda} , \tag{3.3}$$

where ρ and λ are functions of x and y,

$$\lambda(x,y) = [(1-x-y)^2 - 4xy]^{1/2}\,, \qquad \rho(x,y) = 2[1-x-y-\lambda(x,y)]^{-1}\,, \tag{3.4}$$

via

$$\mathrm{Tri}(p_1, p_2+p_3) = \Phi_1(m,v)/(1-v) \tag{3.5}$$

$$\mathrm{Box}(p_1,p_2,p_3) = \Phi_1\left(m^2/(uv), m/(uv)\right)/(uv)\,. \tag{3.6}$$

Their small-mass expansion yields the following expressions for the triangle and the box

$$\mathrm{Tri}(p_1, p_2+p_3) = -\frac{\log m \log v + 2\mathrm{Li}_2(1-v)}{1-v}\,, \tag{3.7}$$

$$\mathrm{Box}(p_1,p_2,p_3) = -\frac{2\log^2 m - 2\log m \log(uv) + \log^2(uv) + 2\zeta_2}{uv}\,, \tag{3.8}$$

where we used the condition (2.3), up to terms vanishing as a power of m. Adding all contributions up, we find

$$\begin{aligned} F_3^{(1)} &= -\log^2\frac{m}{u} - \log^2\frac{m}{v} - \log^2\frac{m}{w} \\ &\quad - \log u \log v - \log v \log w - \log w \log u \\ &\quad - 2\mathrm{Li}_2(1-u) - 2\mathrm{Li}_2(1-v) - 2\mathrm{Li}_2(1-w) - 3\zeta_2\,. \end{aligned} \tag{3.9}$$

It is instructive to compare this results to the conformal case, calculated within dimensional regularization (or rather reduction), [39],

$$\begin{aligned} F_3^{(1)}(\varepsilon) &= -\frac{1}{\varepsilon^2}\left[\left(\frac{-\mu_{\mathrm{DR}}^2}{u}\right)^{\varepsilon} + \left(\frac{-\mu_{\mathrm{DR}}^2}{v}\right)^{\varepsilon} + \left(\frac{-\mu_{\mathrm{DR}}^2}{w}\right)^{\varepsilon}\right] \\ &\quad - \log u \log v - \log v \log w - \log w \log u \\ &\quad - 2\mathrm{Li}_2(1-u) - 2\mathrm{Li}_2(1-v) - 2\mathrm{Li}_2(1-w) + \frac{9}{2}\zeta_2\,. \end{aligned} \tag{3.10}$$

We observe that the finite parts are identical in the two cases, except for the coefficient of ζ_2. When eq. (3.10) expanded in the Laurent series, the coefficient of the double logarithms of $\mu_{\mathrm{DR}}^2/(u,v,w)$ are half of the off-shell case, as anticipated. This is the well-known doubling phenomenon observed back in the early days of QED [43, 44] and well-understood by now as a result of an extra, the so-called ultra-soft, region [45–47] of loop momentum producing leading effects on par with other regimes present in both.

4 Two loops

We now proceed to the two loop calculation. The integrands for the form factor $F_3^{(2)}$ were constructed in ref. [34] using (generalized) unitarity cut technique. With a slight change of the nomenclature compared to [34], the relevant graphs shown in figure 3 generate the

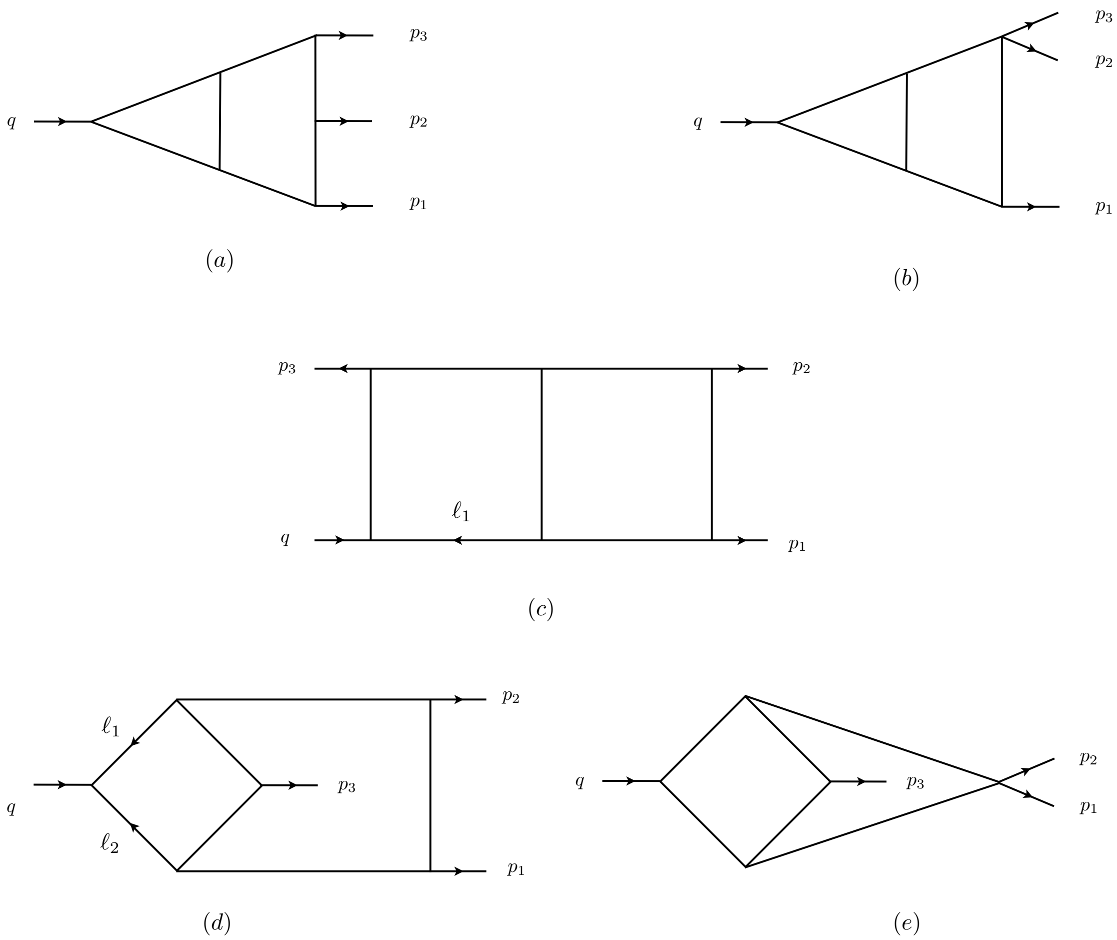

Figure 3. Graphs contributing to the two-loop form factor $F_3^{(2)}$. The integrands, built from product of propagators read off from these diagrams, are accompanied by numerators according to eqs. (4.1)–(4.5).

following integrals[2]

$$\mathrm{TriPent}(p_1,p_2,p_3) = \mathrm{e}^{2\varepsilon\gamma_{\mathrm{E}}} \int \frac{d^D\ell_1}{i\pi^{D/2}} \int \frac{d^D\ell_2}{i\pi^{D/2}} \frac{q^2 s_{12} s_{23}}{\mathrm{denom}_{(a)}}\,, \tag{4.1}$$

$$\mathrm{TriBox}(p_1,p_2+p_3) = \mathrm{e}^{2\varepsilon\gamma_{\mathrm{E}}} \int \frac{d^D\ell_1}{i\pi^{D/2}} \int \frac{d^D\ell_2}{i\pi^{D/2}} \frac{q^2[s_{12}+s_{31}]}{\mathrm{denom}_{(b)}}\,, \tag{4.2}$$

$$\mathrm{DBox}(p_1,p_2,p_3) = \mathrm{e}^{2\varepsilon\gamma_{\mathrm{E}}} \int \frac{d^D\ell_1}{i\pi^{D/2}} \int \frac{d^D\ell_2}{i\pi^{D/2}} \frac{s_{12}[s_{31}\,\ell_1\cdot p_1 - s_{23}\,\ell_1\cdot p_2]}{\mathrm{denom}_{(c)}}\,, \tag{4.3}$$

$$\mathrm{NBox}(p_1,p_2,p_3) = \mathrm{e}^{2\varepsilon\gamma_{\mathrm{E}}} \int \frac{d^D\ell_1}{i\pi^{D/2}} \int \frac{d^D\ell_2}{i\pi^{D/2}} \frac{s_{12}[\frac{1}{2}s_{23}s_{31} - s_{23}\ell_1\cdot p_2 - s_{31}\,\ell_2\cdot p_1]}{\mathrm{denom}_{(d)}}\,, \tag{4.4}$$

$$\mathrm{NTriBox}(p_1+p_2,p_3) = \mathrm{e}^{2\varepsilon\gamma_{\mathrm{E}}} \int \frac{d^D\ell_1}{i\pi^{D/2}} \int \frac{d^D\ell_2}{i\pi^{D/2}} \frac{\frac{1}{2}q^2[s_{23}+s_{31}]}{\mathrm{denom}_{(e)}}\,, \tag{4.5}$$

[2]Here and below, we set the mass scale of dimensional regularization to one, $\mu_{\mathrm{DR}}^2 = 1$.

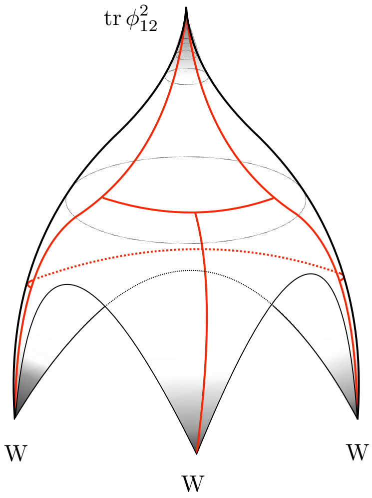

Figure 4. World-sheet perspective of the three-leg form factor and the non-planar graph from figure 3 (d) overlayed on it: it demonstrates why it produces contribution of leading order in color.

where the denominator structure can readily be read off from the corresponding graphs. In terms of these integrals, the two-loop form factor is given by the expression

$$\begin{aligned} F_3^{(2)} = \sum_{n=0}^{2} \mathbb{P}^n \Big[& \mathrm{TriBox}(p_1, p_2+p_3) + \mathrm{TriBox}(p_3, p_1+p_2) + \mathrm{TriPent}(p_1,p_2,p_3) \\ & + \mathrm{DBox}(p_1,p_2,p_3) + \mathrm{DBox}(p_3,p_2,p_1) + \mathrm{NBox}(p_1,p_2,p_3) \\ & + \mathrm{NTriBox}(p_1+p_2,p_3) \Big] \, . \end{aligned} \tag{4.6}$$

Notice that starting from this order, there are non-planar graphs which are leading order in color, i.e., figure 3 (d) and (e). The reason for this is that the operator $\mathrm{tr}\, \phi_{12}^2$ is a singlet with respect to the $\mathrm{SU}(N)$ and thus does not 'participate' in color traces. It becomes quite obvious from the world-sheet perspective of the matrix element (1.3) demonstrated in figure 4 where the operator corresponds to the closed string state, while the W-bosons to the open ones.

Out of all the contributions in eq. (4.6), a truly new integral, which was not addressed in existing literature, is the tri-pentagon, figure 3 (a). So we start with its analysis first in the next section.

4.1 Tri-pentagon

Let us begin with the construction of the canonical basis for the tri-pentagon family, see figure 3 (a), by routing the loop momenta ℓ_1 and ℓ_2 according to the following definitions of propagator denominators D_i $(i = 1, \dots, 7)$ and irreducible scalar products D_8 and D_9,

$$\begin{aligned} &D_1 = -\ell_1^2 , \quad D_2 = -(\ell_1+p_1)^2 , \quad D_3 = -(\ell_1+p_1+p_2)^2 , \qquad D_4 = -(\ell_1+p_1+p_2+p_3)^2 , \\ &D_5 = -\ell_2^2 , \quad D_6 = -(\ell_2-\ell_1)^2 , \quad D_7 = -(\ell_2+p_1+p_2+p_3)^2 , \quad D_8 = -(\ell_2+p_1)^2 , \\ &\qquad\qquad\qquad\qquad\qquad\quad\;\; D_9 = -(\ell_2+p_1+p_2)^2 , \end{aligned} \tag{4.7}$$

such that

$$G_{a_1a_2a_3a_4a_5a_6a_7a_8a_9} \equiv \mathrm{e}^{2\varepsilon\gamma_{\rm E}} \int \frac{d^D\ell_1}{i\pi^{D/2}} \int \frac{d^D\ell_2}{i\pi^{D/2}} \prod_{i=1}^{9} D_i^{-a_1} \,. \tag{4.8}$$

The use of dimensionally regularized integrals is required for proper use of the integration-by-part technique in order to work with vanishing surface loop integrals, which are at the heart of the formalism [48]. An IBP reduction with the `FIRE` code [49–51] immediately reveals 49 initial Master Integrals (MIs), which are further reduced to 46 by finding equivalences among them with the `LiteRed` software [52], generating thus the primary basis

$$\begin{aligned}
\boldsymbol{I} = \{ & G_{000111000}, G_{001011000}, G_{010001100}, G_{010011000}, G_{001011100}, G_{001011200}, \\
& G_{001111000}, G_{001112000}, G_{010011100}, G_{010011200}, G_{010110100}, G_{010111000}, \\
& G_{010112000}, G_{011001100}, G_{011001200}, G_{011011000}, G_{011012000}, G_{100110100}, \\
& G_{101001100}, G_{101001200}, G_{101010100}, G_{110001100}, G_{110001200}, G_{110010100}, \\
& G_{001111100}, G_{010111100}, G_{011011100}, G_{011011200}, G_{011012100}, G_{011012200}, \\
& G_{011021100}, G_{012011100}, G_{021011100}, G_{011110100}, G_{011111000}, G_{011112000}, \\
& G_{101011100}, G_{101110100}, G_{110011100}, G_{110110100}, G_{111001100}, G_{111001200}, \\
& G_{111010100}, G_{011111100}, G_{111011100}, G_{111110100} \} \,.
\end{aligned} \tag{4.9}$$

Next we turn to the derivation of differential equations for $\boldsymbol{I}$ making use of a `FIRE` interface to `LiteRed`,

$$\partial_i \boldsymbol{I} = \boldsymbol{M}_i \cdot \boldsymbol{I} \,, \tag{4.10}$$

in the kinematical invariants $i = u, v, w, m$. The goal is now to convert them to the canonical form [53]

$$\partial_i \boldsymbol{J} = \varepsilon \boldsymbol{A}_i \cdot \boldsymbol{J} \,, \qquad \varepsilon \boldsymbol{A}_i = \boldsymbol{T}^{-1} \cdot \boldsymbol{M} \cdot \boldsymbol{T} - \boldsymbol{T}^{-1} \cdot \partial_i \boldsymbol{T} \,, \tag{4.11}$$

with some transformation matrix $\boldsymbol{T}$. In fact, what we need is an *asymptotically* canonical basis, which captures all logarithmically enhanced and constant terms in m as m goes to zero. This can easily be accomplished by keeping track of only singular power-like terms in the 'virtuality' matrix

$$\boldsymbol{A}_m = \frac{\varepsilon}{m} \boldsymbol{A}_m^0 + O(m^0) \,, \tag{4.12}$$

as was explained at length in ref. [54].

Splitting the basis elements of $\boldsymbol{I}$ into sectors, we form their linear combinations accompanied by unknown functions of the Mandelstam-like variables (u, v, w) and fix the former by enforcing the ε-form of the differential equations (4.11). Having fixed the diagonal blocks in this manner, the off-diagonal ones can be constrained by using two available software packages `Canonica` [55, 56] and `Libra` [57, 58]. To achieve this, one first transforms the equations to the Fuchsian, i.e., dLog, form followed by factorization of the ε-dependence into an overall factor [57]. `Canonica` is solely based on built-in Mathematica commands and fails to successfully solve corresponding systems of linear equations. Therefore, we used two strategies in our analysis. One was based exclusively on `Libra`. However, having constructed canonical form of differential equations, we discovered that five of its elements did not possess uniform transcendentality[3] (UT), namely, J_i's with indices $i = 34, 43, 44, 45, 46$. So in our attempt to alleviate this problem, we deduced yet another form of the canonical differential equations by the combined use of `Canonica` (to bring equations to the Fuchsian form) and `Libra` (for the derivation of the ε-form). Though, the basis found was slightly different from the first one, nevertheless the very same five elements suffered from the very same problem. Obviously, this was not in any way an obstruction in our subsequent steps of solving theses 'canonical' equations rather it was merely a nuisance: instead of fixing a set of integration constant of uniform transcendentality at each ε^n-order, we had to use a sum of constants of increasing transcendental weight $w_i \leq n$. The asymptotically canonical basis, which we used in the explicit iterative solution of the differential equations, is

$$J_1 = \varepsilon^2(3m - u - v - w)G_{000122000}\,, \tag{4.13}$$
$$J_2 = \varepsilon^2 u G_{001022000}\,, \tag{4.14}$$
$$J_3 = \varepsilon^2 v G_{010002200}\,, \tag{4.15}$$
$$J_4 = \varepsilon^2 m G_{010022000}\,, \tag{4.16}$$
$$J_5 = \varepsilon^3(v + w)G_{002011100}\,,\,, \tag{4.17}$$
$$J_6 = \varepsilon^2(u + v + w)((2\varepsilon - 1)G_{001011200} + \varepsilon G_{002011100})\,, \tag{4.18}$$
$$J_7 = \varepsilon^2(2\varepsilon - 1)m G_{002111000}\,, \tag{4.19}$$
$$J_8 = \varepsilon^3(v + w)G_{001112000}\,, \tag{4.20}$$
$$J_9 = \varepsilon^3(u + w)G_{020011100}\,, \tag{4.21}$$
$$J_{10} = \varepsilon^2(1 - 2\varepsilon)(u + v + w)G_{010011200}\,, \tag{4.22}$$
$$J_{11} = \frac{7}{25}\varepsilon^2(1 - 2\varepsilon)^2 G_{010110100}\,, \tag{4.23}$$
$$J_{12} = \varepsilon^2(2\varepsilon - 1)(3\varepsilon - 1)G_{010111000}\,, \tag{4.24}$$
$$J_{13} = \varepsilon^3(u + w)G_{010112000}\,, \tag{4.25}$$
$$J_{14} = \varepsilon^2(2\varepsilon - 1)(3\varepsilon - 1)G_{011001100}\,, \tag{4.26}$$
$$J_{15} = \varepsilon^3 v G_{011001200}\,, \tag{4.27}$$
$$J_{16} = \varepsilon^2(2\varepsilon - 1)(3\varepsilon - 1)G_{011011000}\,, \tag{4.28}$$
$$J_{17} = \varepsilon^3 u G_{011012000}\,, \tag{4.29}$$
$$J_{18} = \frac{1}{25}\varepsilon^2(1 - 2\varepsilon)^2 G_{100110100}\,, \tag{4.30}$$
$$J_{19} = \varepsilon^2(2\varepsilon - 1)(3\varepsilon - 1)G_{101001100}\,, \tag{4.31}$$
$$J_{20} = \varepsilon^3(v + w)G_{101001200}\,, \tag{4.32}$$
$$J_{21} = \frac{7}{25}\varepsilon^2(1 - 2\varepsilon)^2 G_{101010100}\,, \tag{4.33}$$
$$J_{22} = \varepsilon^2(1 - 2\varepsilon)m G_{210001100}\,, \tag{4.34}$$

[3] We would like to thank Johannes Henn for instructive communications on this point.

$$J_{23} = \varepsilon^3(u+w)G_{110001200}\,, \tag{4.35}$$

$$J_{24} = \frac{7}{25}\varepsilon^2(1-2\varepsilon)^2 G_{110010100}\,, \tag{4.36}$$

$$J_{25} = \varepsilon^4(v+w)G_{001111100}\,, \tag{4.37}$$

$$J_{26} = \varepsilon^4(u+w)G_{010111100}\,, \tag{4.38}$$

$$J_{27} = \varepsilon^4(u+v)G_{011011100}\,, \tag{4.39}$$

$$\begin{aligned} J_{28} = \frac{1}{2}\varepsilon^2\big[&-vG_{010002200} + mG_{010022000} \\ &+ 2\varepsilon v((u+v+w)G_{011011200} - 2G_{011001200})\big]\,, \end{aligned} \tag{4.40}$$

$$J_{29} = \frac{1}{2}\varepsilon^2\big[-2vG_{010002200} - 11mG_{010022000} + 2\varepsilon uvG_{011012100}\big]\,, \tag{4.41}$$

$$\begin{aligned} J_{30} = \frac{\varepsilon^2}{v+w}\big[&-uvG_{010002200} - m(u-2v-2w)G_{010022000} \\ &+ u\left[2\left(6\varepsilon^2 - 5\varepsilon + 1\right)G_{011001100} - 4\varepsilon vG_{011001200} + mv(u+v+w)G_{011012200}\right]\big]\,, \end{aligned} \tag{4.42}$$

$$J_{31} = \frac{1}{2}\varepsilon^2\big[vG_{010002200} + 12mG_{010022000} + 2\varepsilon u(u+v+w)G_{011021100}\big]\,, \tag{4.43}$$

$$\begin{aligned} J_{32} = \frac{1}{4}\varepsilon^2\big[&vG_{010002200} + 5mG_{010022000} \\ &+ 4\big[\left(-6\varepsilon^2 + 5\varepsilon - 1\right)G_{011001100} + \varepsilon vG_{011001200} + \varepsilon m(v+w)G_{012011100}\big]\big]\,, \end{aligned} \tag{4.44}$$

$$\begin{aligned} J_{33} = -\frac{1}{4}\varepsilon^2\big[&3vG_{010002200} + 17mG_{010022000} \\ &+ 4\left(6\varepsilon^2 - 5\varepsilon + 1\right)G_{011001100} - 4\varepsilon m(u+w)G_{021011100}\big]\,, \end{aligned} \tag{4.45}$$

$$J_{34} = \frac{1}{5}\varepsilon^2\big[\left(14\varepsilon^2 - 9\varepsilon + 1\right)(2G_{010110100} - 4G_{110010100}) + 5\varepsilon vG_{011110100}\big]\,, \tag{4.46}$$

$$J_{35} = (1-2\varepsilon)\varepsilon^3 vG_{011111000}\,, \tag{4.47}$$

$$J_{36} = \varepsilon^3 uvG_{011112000}\,, \tag{4.48}$$

$$J_{37} = \varepsilon^4(v+w)G_{101011100}\,, \tag{4.49}$$

$$J_{38} = -\frac{7}{5}\varepsilon^3(2\varepsilon-1)(v+w)G_{101110100}\,, \tag{4.50}$$

$$J_{39} = \varepsilon^4(u+w)G_{110011100}\,, \tag{4.51}$$

$$J_{40} = -\frac{7}{5}\varepsilon^3(2\varepsilon-1)(u+w)G_{110110100}\,, \tag{4.52}$$

$$J_{41} = \varepsilon^3(1-2\varepsilon)uG_{111001100}\,, \tag{4.53}$$

$$J_{42} = \varepsilon^3 uvG_{111001200}\,, \tag{4.54}$$

$$J_{43} = \frac{1}{5}\varepsilon^2\big[2\left(14\varepsilon^2 - 9\varepsilon + 1\right)(2G_{101010100} - 4G_{110010100}) + 5\varepsilon uG_{111010100}\big]\,, \tag{4.55}$$

$$\begin{aligned} J_{44} = \frac{1}{7}\varepsilon^2\big[&7\varepsilon^2 v(v+w)G_{011111100} + 2\varepsilon(7\varepsilon-1)vG_{011110100} \\ &+ 2(2\varepsilon-1)(7\varepsilon-1)(2G_{010110100} - 4G_{110010100})\big]\,, \end{aligned} \tag{4.56}$$

$$\begin{aligned} J_{45} = \frac{1}{7}\varepsilon^2\big[&\left(14\varepsilon^2 - 9\varepsilon + 1\right)(2G_{101010100} - 4G_{110010100}) \\ &+ \varepsilon u\big[2(7\varepsilon-1)G_{111010100} + 7\varepsilon(u+w)G_{111011100}\big]\big]\,, \end{aligned} \tag{4.57}$$

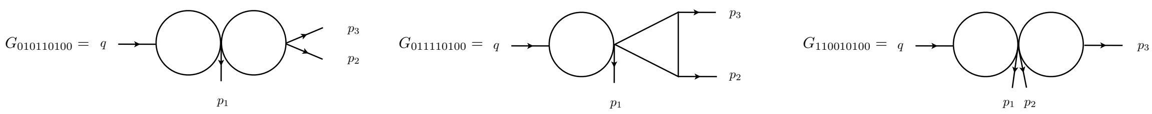

Figure 5. Diagrammatic form of the integrals forming the element J_{34} of the canonical basis (4.13).

$$J_{46} = \frac{1}{5}\varepsilon^2 \big[(2\varepsilon-1)(7\varepsilon-1)\big[2G_{010110100} + 2G_{101010100} - 8G_{110010100}\big] + 2\varepsilon(7\varepsilon-1)\big[vG_{011110100} + uG_{111010100}\big] - 7\varepsilon(2\varepsilon-1)uvG_{111110100}\big]\,. \tag{4.58}$$

First, we solved the 'virtuality' differential equation, in the small-virtuality limit

$$\boldsymbol{J} = m^{\varepsilon \boldsymbol{A}_m^0} \cdot \boldsymbol{J}_0 \tag{4.59}$$

related to the 'massless' MIs $\boldsymbol{J}_0$ via the matrix exponent $m^{\varepsilon \boldsymbol{A}_m^0}$. Next, we solved the $m = 0$ limit of the differential equations in Mandelstam-like variables via the Chen iterated integrals on a piece-wise contour [59]

$$\boldsymbol{J}_0 = P_\gamma \exp\left(\varepsilon \int_{[0,u]\cup[0,v]\cup[0,w]} \boldsymbol{A}^0\right) \boldsymbol{J}_{00}\,, \tag{4.60}$$

with the differential of the A-matrices $\boldsymbol{A} = du\boldsymbol{A}_u^0 + dv\boldsymbol{A}_v^0 + dw\boldsymbol{A}_w^0$. At each order of the ε-expansion, we found solutions in terms of multiple polylogarithms [60].

Finally, we had to fix the vector of the integration constants $\boldsymbol{J}_{00}$ at each order of the ε-expansion

$$\boldsymbol{J}_{00} = \sum_{p\geq 0} \varepsilon^p \boldsymbol{c}^{(p)}\,. \tag{4.61}$$

To accomplish this, we used two criteria: (i) the absence of spurious poles in the right-hand sides of differential equations at the location of $u+v$, $v+w$ and $w+u$ poles and (ii) numerical integration with `FIESTA` [61] with subsequent use of the `PSLQ` algorithm [62]. However, these considerations alone did no allow us to fully analytically determine all of the integration constants. We needed further input. We found that all undetermined contributions are reduced a set of unknowns which can be determined in turn by evaluating one of the elements of the canonical basis explicitly. The element in question is J_{34}, which is given by a linear combination of factorized products of bubbles and triangles, eq. (4.46), as demonstrated in figure 5. This can be easily calculated making use of the code `MBcreate.m` [63]. It yielded the following expressions

$$G_{010110100} = v^{-\varepsilon}(u+v+w)^{-\varepsilon}\frac{e^{2\gamma\varepsilon}\,\Gamma(1-\varepsilon)^4\Gamma(\varepsilon)^2}{\Gamma(2-2\varepsilon)^2}\,, \tag{4.62}$$

$$G_{011110100} = m^{-2\varepsilon}v^{-\varepsilon-1}(u+v+w)^{-\varepsilon}\frac{e^{2\gamma\varepsilon}\,\Gamma(1-\varepsilon)^2\Gamma(\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(2-2\varepsilon)} \times \big[m^{2\varepsilon}\Gamma(-\varepsilon)^2\Gamma(\varepsilon+1) + v^\varepsilon\Gamma(1-\varepsilon)\Gamma(\varepsilon)\left(2m^\varepsilon\Gamma(-\varepsilon) + v^\varepsilon\Gamma(1-2\varepsilon)\Gamma(\varepsilon)\right)\big]\,, \tag{4.63}$$

$$G_{110010100} = m^{-\varepsilon}(u+v+w)^{-\varepsilon}\frac{e^{2\gamma\varepsilon}\,\Gamma(1-\varepsilon)^4\Gamma(\varepsilon)^2}{\Gamma(2-2\varepsilon)^2}\,. \tag{4.64}$$

Matching their expansions to the iterative solution, we found our final result. The expressions are too lengthy to be displayed here in the body or appendices, so they are relegated to the accompanying Mathematica notebook `TriPentagonA2Z.nb` in the supplementary material, where an interested reader could find as well all steps from-A-to-Z for the determination of their expressions starting with necessary initial IBP reductions. The tri-pentagon (4.1) is then given by the integral

$$\mathrm{TriPent} = uv(u+v+w)G_{111111100}\,, \tag{4.65}$$

which is not one of the elements of the above basis, but can be easily reduced to them by means of an IBP reduction. The latter gives

$$\begin{aligned}\mathrm{TriPent} = \frac{1}{\varepsilon^4}\bigg[&-J_3-\frac{11}{2}J_4-\frac{2(1-7\varepsilon)(u+v+w)}{7(v+w)}[5J_{11}-J_{34}]\\ &-\frac{2(1-7\varepsilon)(u+v+w)}{7(u+w)}[5J_{21}-J_{43}]\\ &+\frac{20(1-7\varepsilon)(u^2+2uv+v^2+3uw+3vw+2w^2)}{7(u+w)(v+w)}J_{24}\\ &-J_{29}+J_{36}+J_{42}+\frac{u}{v+w}J_{44}+\frac{v}{u+w}J_{45}+\frac{5}{7}J_{46}\bigg]\,.\end{aligned} \tag{4.66}$$

Notice that some of the MIs in this expression do not possess UT individually, however, in the sum TriPent is indeed UT. However, it is not a pure function: multiple polylogarithms are accompanied by rational prefactors of the (u,v,w) variables.

4.2 Tri-box

The tri-box graph in figure 3 (b) is related to the Davydychev-Ussyukina function Φ_2 given in eq. (3.3),

$$\mathrm{TriBox}(p_1,p_2+p_3) = (u+w)\Phi_2(m,v)\,. \tag{4.67}$$

Its small-m expansion immediately produces the sought after expression

$$\begin{aligned}\mathrm{TriBox}(p_1,p_2+p_3) &= \left[\frac{1}{2}\mathrm{Li}_2\left(\frac{u}{u-1}\right)+\frac{1}{4}\log^2\frac{1-u}{u}+\frac{\zeta_2}{2}\right]\log^2 m\\ &+\left[3\mathrm{Li}_3\left(\frac{u}{u-1}\right)-\mathrm{Li}_2\left(\frac{u}{u-1}\right)\log u-\frac{1}{2}\log(1-u)\log^2\frac{1-u}{u}-\zeta_2\log\frac{(1-u)^3}{u^2}\right]\log m\\ &+6\mathrm{Li}_4\left(\frac{u}{u-1}\right)-3\mathrm{Li}_3\left(\frac{u}{u-1}\right)\log(u)+\frac{1}{2}\mathrm{Li}_2\left(\frac{u}{u-1}\right)\log^2 u\\ &+\frac{1}{4}\log^2(1-u)\log^2\frac{1-u}{u}+\frac{1}{2}\zeta_2\log^2 u+3\zeta_2\log(1-u)\log\frac{1-u}{u}+\frac{21}{2}\zeta_4\,.\end{aligned} \tag{4.68}$$

4.3 Double boxes and nonplanar tri-box

The non- and planar double boxes in the near-off-shell kinematics were calculated recently in ref. [54]. The asymptotically canonical bases for the families of graphs in figure 3 (c) and (d)

consist of 62 and 97 elements, respectively. Thus, all we need to do in order to evaluate the integrals in eqs. (4.3) and (4.4) is to perform their IBP reduction to the canonical elements constructed in [54]. This task is elementary making use of the `FIRE` code and we found

$$\text{DBox} = -\frac{1}{2\varepsilon^4}\left[\frac{w}{v}J_{54} + \frac{v}{u+w}J_{55} - J_{59} - J_{61}\right], \tag{4.69}$$

and

$$\begin{aligned}
\text{NBox} = -\frac{1}{4\varepsilon^4}\Bigg[&\frac{1585645}{22176}J_1 - \frac{949}{528}J_2 + \frac{72337}{1848}J_3 - \frac{4403017}{22176}J_4 + \frac{7045}{528}J_5 - \frac{1153}{88}J_8 \\
&- \frac{1715}{88}J_9 - \frac{589}{44}J_{11} + \frac{3406615}{5544}J_{12} - \frac{58561}{11088}J_{13} - \frac{84}{11}J_{14} - \frac{1/22}{J}{}_{15} + \frac{6794}{63}J_{17} \\
&+ \frac{8495}{126}J_{18} - \frac{688}{21}J_{19} - \frac{12673}{3696}J_{20} - \frac{42541}{308}J_{21} - \frac{64103}{3696}J_{22} - \frac{2}{3}J_{23} - \frac{793}{44}J_{24} \\
&- \frac{78637}{308}J_{25} + \frac{55897}{1848}J_{26} + \frac{5064085}{22176}J_{27} + \frac{799}{352}J_{28} - \frac{42905}{7392}J_{29} + J_{33} - 8J_{34} \\
&- \frac{5}{2}J_{35} + \frac{7}{2}J_{36} + 6J_{37} - 4J_{38} + 3J_{39} + 15J_{40} - 2J_{41} + \frac{1}{6}J_{43} - \frac{17}{6}J_{44} - \frac{71}{21}J_{45} \\
&- \frac{82}{7}J_{46} - J_{47} + \frac{788}{21}J_{48} + \frac{464}{21}J_{49} + 8J_{50} - 2J_{54} + \frac{5}{2}J_{55} - \frac{13}{2}J_{56} - \frac{176}{21}J_{57} \\
&- \frac{281}{21}J_{58} - \frac{667}{21}J_{59} + \frac{352}{21}J_{60} - 2J_{61} - 2J_{62} + 6J_{63} + 11J_{64} + J_{65} + 7J_{66} \\
&- J_{67} + 6J_{68} + 2J_{69} + 4J_{70} - \frac{122711}{924}J_{71} - \frac{60253}{462}J_{72} + \frac{271}{132}J_{73} - \frac{1}{3}J_{74} \\
&- \frac{82953}{308}J_{75} + \frac{118609}{462}J_{76} - \frac{17}{44}J_{77} - \frac{2w}{v}J_{78} + \frac{2v}{u+w}J_{79} + \frac{2w}{u+v}J_{80} \\
&- \frac{2(v+w)}{w}J_{81} - \frac{3}{11}J_{82} + \frac{3}{22}J_{83} - \frac{25}{22}J_{84} + \frac{25}{22}J_{85} - 4J_{86} + \frac{25}{22}J_{87} \\
&+ \frac{8(u+v+w)}{v+w}J_{88} + \frac{21}{22}J_{89} + \frac{1}{44}J_{90} - \frac{25}{132}J_{91} + \frac{4}{3}J_{92} - \frac{25}{66}J_{93} + \frac{25}{132}J_{94} - J_{95} + J_{96}\Bigg],
\end{aligned} \tag{4.70}$$

for the planar and non-planar graphs in figures 3 (c) and (d), respectively. These are way too lengthy to be presented in the explicit form in the body of the paper. Therefore, for the reader's convenience, we spell them out in the `Mathematica` notebook `Integrals.nb` attached in the supplementary material.

Finally, the nonplanar tri-box in figure (3) (e) is just one of the MIs in the nonplanar doublebox basis, namely,

$$\text{NTriBox} = \frac{2(u+v+w)}{\varepsilon^4(v+w)}J_{88}\,. \tag{4.71}$$

Of course, this graph was calculated in ref. [64], where it was found into factorize after a Fourier transform to the square of Φ_1:

$$\text{NTriBox} = \frac{1-u}{2}[\Phi_1(m,u)]^2\,. \tag{4.72}$$

We indeed confirmed our agreement with it on the constraint (2.3), $u+v+w = 1 + O(m)$. This concludes our calculation of contributing two-loop graphs. All of the integrals reported in this section are UT, however, none are pure.

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5 Adding things up

Finally, we are in a position to add up all of the calculated integrals.

5.1 Infrared exponentiation and general structure

As we alluded to in the introduction, we anticipate [25–27] that the infrared logarithms, i.e., $\log m$, exponentiate such that the form factor takes the form

$$\log F_3 = -\frac{\Gamma_{\text{oct}}(g)}{4}\left[\log^2\left(\frac{m}{u}\right)+\log^2\left(\frac{m}{v}\right)+\log^2\left(\frac{m}{w}\right)\right]+\text{Fin}_3\left(u,v,w;g\right)+O(m^2)\,, \quad (5.1)$$

with Γ_{oct} being the octagon anomalous dimension [28–30] and Fin_3 being a finite part: it depends only on scalar products of momenta of external states and the 't Hooft coupling constant g. It also depends on the type of the operator insertion in (1.3) as well as helicities of external states. Fin_3 develops a perturbative expansion

$$\text{Fin}_3 = g^2 f_3^{(1)} + g^4 f_3^{(2)} + \ldots\,. \quad (5.2)$$

The infrared exponent Γ_{oct} is known exactly to all orders in the coupling g and is given by [29]:

$$\Gamma_{\text{oct}}(g) = -\frac{2}{\pi^2}\log\cosh\left(2\pi g\right) = 4g^2 - 16\zeta_2 g^4 + \ldots\,. \quad (5.3)$$

Here, we expanded it to the first two orders, relevant for our current study. We would like to point out the absence in eq. (5.1) of linear powers in $\log m$ in contrast to the kinematical regime considered in refs. [1, 15–17], where all external particles' momenta were strictly massless and states propagating in loops' perimeters where taken massive:[4] there is no analogue of the collinear anomalous dimension in the off-shell regime!

The expansion of $\log F_3$ in powers of g is given by

$$\log F_3 = g^2 F_3^{(1)} + g^4\left(F_3^{(2)} - \frac{1}{2}[F_3^{(1)}]^2\right) + \ldots\,, \quad (5.4)$$

and can be matched onto the expressions for $F_3^{(1)}$ and $F_3^{(2)}$ in terms of scalar integrals given by (3.1) and (4.6), respectively. Focusing on the infrared divergent part first, we combine the integrals computed above to find

$$\log F_3\Big|_{\text{div}} = [-3g^2 + 12\zeta_2 g^4 + \ldots]\log^2 m + [2g^2 - 8\zeta_2 g^2 + \ldots]\log m\log(uvw)\,, \quad (5.5)$$

in full agreement with our expectation (5.1).

Several comments are in order. Individual two-loop integrals in (4.6) contain $\log^4 m$ as well $\log^3 m$ terms. They cancel, however, in the difference between $F_3^{(2)}$ and the square of one-loop form factor $F_3^{(1)}$ in the $O(g^4)$ coefficient in (5.4). Individual two-loop integrals, i.e., coefficients accompanying the powers of $\log m$, are, in general, expressed in terms of multiple polylogarithms [60]. As can be seen in attached `Mathematica` notebook `Integrals.nb` (see the supplementary material), the coefficients of $\log^2 m$ and $\log m$ in (5.5) are determined

[4] The relation (5.1) is, strictly speaking, a conjecture supported by an array of explicit computations [25–27] as well as a general intuition about IR properties of gauge theories [19–21].

solely by ordinary logarithms. To observe the cancellations of higher powers of the infrared logarithms as well as simplifications of $\log^2 m$ and $\log m$ terms in (5.5) we used a combination of the symbol map [65] along with high-precision numerical computations offered by the `GiNaC` integrator [66] through the interactive `Ginsh` environment of the `PolyLogTools` package [67]. As we emphasized in earlier sections, individual two-loop integrals are not pure UT functions. They, however, do neatly combine into a pure UT expression when collected together in $F_3^{(2)}$.

5.2 Finite part

Let us now move on to the finite part Fin_3. From eq. (3.1) it is easy to see that at one loop we have

$$\begin{aligned} f_3^{(1)}(u,v,w) =& -\log u \log v - \log v \log w - \log w \log u \\ & - 2\mathrm{Li}_2(1-u) - 2\mathrm{Li}_2(1-v) - 2\mathrm{Li}_2(1-w) - 3\zeta_2\,. \end{aligned} \tag{5.6}$$

The two-loop finite part $f_3^{(2)}$ is given by the $\log m$-free term of the $O(g^4)$ coefficient[5] in (5.4). It is a complicated combination of multiple polylogarithms of weight 4. On the route to simplify this expression, it is instructive to consider its symbol map first. Using the `PolyLogTools`, we found out that the symbol of $f_3^{(3)}$ is given by

$$\begin{aligned} \mathcal{S}[f_3^{(3)}] =& -2u\otimes(1-u)\otimes(1-u)\otimes\frac{1-u}{u} + u\otimes(1-u)\otimes u\otimes\frac{1-u}{u} \\ & - u\otimes(1-u)\otimes v\otimes\frac{1-v}{v} - u\otimes(1-u)\otimes w\otimes\frac{1-w}{w} \\ & - u\otimes v\otimes(1-u)\otimes\frac{1-v}{v} - u\otimes v\otimes(1-v)\otimes\frac{1-u}{u} \\ & + u\otimes v\otimes w\otimes\frac{1-u}{u} + u\otimes v\otimes w\otimes\frac{1-v}{v} \\ & + u\otimes v\otimes w\otimes\frac{1-w}{w} - u\otimes w\otimes(1-u)\otimes\frac{1-w}{w} \\ & + u\otimes w\otimes v\otimes\frac{1-u}{u} + u\otimes w\otimes v\otimes\frac{1-v}{v} \\ & + u\otimes w\otimes v\otimes\frac{1-w}{w} - u\otimes w\otimes(1-w)\otimes\frac{1-u}{u} \\ & + \text{ cyclic permutations}\,. \end{aligned} \tag{5.7}$$

This symbol is identical to the symbol of a local function of the following combination of logarithms and classical polylogarithms:

$$\begin{aligned} R_3^{(2)}(u,v,w) =& -2\left[\mathrm{J}\left(-\frac{uv}{w}\right) + \mathrm{J}\left(-\frac{vw}{u}\right) + \mathrm{J}\left(-\frac{wu}{v}\right)\right] - 8\sum_{i=1}^{3}\left(\mathrm{Li}_4\left(1-u_i^{-1}\right) + \frac{\log^4 u_i}{4!}\right) \\ & - 2\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i^{-1})\right)^2 + \frac{1}{2}\left(\sum_{i=1}^{3}\log^2 u_i\right)^2 - \frac{\log^4(uvw)}{4!}\,, \end{aligned} \tag{5.8}$$

[5] Less $4\zeta_2\left(\log^2(u)+\log^2(v)+\log^2(w)\right)$ due to our definition of the divergent part which includes a finite term as well.

with the $\mathrm{J}(z)$ function defined as

$$\mathrm{J}(z) = \mathrm{Li}_4(z) - \log(-z)\mathrm{Li}_3(z) + \frac{\log^2(-z)}{2!}\mathrm{Li}_2(z) - \frac{\log^3(-z)}{3!}\mathrm{Li}_1(z) - \frac{\log^4(-z)}{48} . \tag{5.9}$$

Here for brevity of the presentation, we employed the set of variables $u_1 \equiv u$, $u_2 \equiv v$ and $u_3 \equiv w$. The $R_3^{(2)}$ function was first uncovered in the computation of the finite part of the three-gluon form factor in the conformal regime [34], i.e., at the origin of the moduli space of $\mathcal{N} = 4$ sYM. However, numerical evaluations of $f_3^{(2)}$ and $R_3^{(2)}$ in several kinematical points clearly indicate that they are different and the difference is not a constant. This is not surprising given that the symbol map is blind to terms such as $\pi^2 \times \mathrm{function}(u, v, w)$. We have constructed an ansatz of all possible terms[6] of the form $\pi^2 \times \{\log(x_i)\log(x_j), \mathrm{Li}_2(x_i), \pi^2\}$ with rational coefficients plus $R_3^{(2)}$. The values of x_i were taken from the following list

$$\left\{u, v, w, 1-u, 1-v, 1-w, 1-\frac{1}{u}, 1-\frac{1}{v}, 1-\frac{1}{w}, -\frac{uv}{w}, -\frac{vw}{u}, -\frac{wu}{v}\right\} . \tag{5.10}$$

Evaluating numerically our ansatz and $f_3^{(2)}$ in several kinematical points using the `Ginsh` integrator allowed us to unambiguously fix these coefficients, and we arrived at

$$\begin{aligned} f_3^{(2)}(u, v, w) = R_3^{(2)}(u, v, w) &+ 3\zeta_2 \left[\log(u)\log(v) + \log(v)\log(w) + \log(w)\log(u)\right] \\ &- 4\zeta_2 \sum_{i=1}^{3} \mathrm{Li}_2\left(1 - u_i^{-1}\right) + 9\zeta_2 \sum_{i=1}^{3} \log^2 u_i + \frac{63\zeta_4}{4} . \end{aligned} \tag{5.11}$$

This concludes our calculation of the finite part at the two-loop order. We see that it is a pure function of uniform transcendentality just as in the conformal case.

5.3 Iterative structure

In the massless case of scattering amplitudes, it became customary to split results according to the so called BDS ansatz [18] and a finite remainder [68, 69]. The same decomposition was established for the case of form factors as well [34]. Such a decomposition admits the following generic from

$$F_3^{(2)} = \frac{1}{2}\big[F_3^{(1)}\big]^2 + 4\zeta_2 \widetilde{F}_3^{(1)} + \mathcal{R}_3^{(2)} . \tag{5.12}$$

In the massless case, $\widetilde{F}_3^{(1)}$ was found to enjoy a very powerful feature, namely, it was determined at two loops to be merely given by the one-loop form factor [34]

$$\widetilde{F}_3^{(1)} = \frac{1}{4} F_3^{(1)}(2\varepsilon) , \tag{5.13}$$

where the factor of $\frac{1}{4}$ is introduced to accommodate the change from Γ_{oct} to Γ_{cusp} of the massless case. This is the well-known cross-order relation [18] encoding the iterative structure of massless amplitudes. It was also confirmed on the Coulomb branch where the external legs were kept massless [16].

[6] Taking into account cyclic symmetry as well as functional relations between Li_2 reduces the number of terms in the ansatz quite significantly.

Adopting the same nomenclature in the current 'off-shell' case, we find that $\widetilde{F}_3^{(1)}$ possesses all of the building blocks of the one-loop form factor $F_3^{(1)}$ but is not directly related to it except for the infrared-divergent terms. It has the form

$$
\begin{aligned}
\widetilde{F}_3^{(1)} &= 3\log^2 m - 2\log m \log uvw \\
&\quad + \frac{3}{4}\left[\log^2 u + \log^2 v + \log^2 w + \log u \log v + \log u \log w + \log v \log w\right] \\
&\quad + \mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w)\,,
\end{aligned}
\tag{5.14}
$$

cf. eq. (5.5), such that $\widetilde{F}_3^{(1)}|_{\mathrm{div}} = F_3^{(1)}|_{\mathrm{div}}$. With this convention, the 'off-shell' remainder function $\mathcal{R}_3^{(2)}$ is related by a constant shift

$$
\mathcal{R}_3^{(2)}(u,v,w) = R_3^{(2)}(u,v,w) + \frac{63\zeta_4}{4}\,. \tag{5.15}
$$

to the one of the conformal case,[7] $R_3^{(2)}$ [34]! Indeed, we could enforce the same iterative structure of the 'off-shell' form factor as in the conformal case at the expense of changing the remainder function $\mathcal{R}_3^{(2)}$.

6 Conclusion

With this paper, we continued our excursion into the land of the Coulomb branch away from the origin in its moduli space. The object under our study was form factor of the lowest component of the stress-tensor multiplet for three massive W-bosons. We were particularly interested in the asymptotic region of their vanishing masses, $m \to 0$. In this case, the emerging infrared divergences are encoded by the logarithms of m, which replace inverse powers of ε in dimensional regularization. However, this is not to be confused with another use of the Coulomb branch advocated in ref. [1], as a means to make amplitudes and form factors finite by giving vacuum expectation values to scalars propagating around quantum loops perimeters. In the latter case, it was established that amplitudes and Sudakov form factors echo the well-known infrared behavior of massless scattering amplitudes and form factors with the infrared physics driven by the cusp anomalous dimension. In counter-distinction, we find instead, that like in the case of scattering amplitudes of four- [24] and five W-bosons [25] and the Sudakov form factor of two W-bosons [26, 27], the infrared logarithms are accompanied by a completely different function of the coupling, the octagon anomalous dimension [28–30]. This reconfirms the role of the latter as the critical infrared exponent of the off-shell kinematics.

Further, the form factor of three W-bosons possesses a nontrivial remainder function. After a proper subtraction of infrared logarithms with judiciously-chosen finite parts, we found it to be identical to the one in the massless case (up to a constant), i.e., the origin of the moduli space. The structure of the collinear limit is however quite different in the two cases. While the massless case inherits its iterative structure in terms of one-loop amplitude/form factor, the case of massive W-bosons is trickier. In order to put it on a firmer foundation, analysis of the five-W amplitude at generic values of Mandelstam-like variables needs to be studied, as opposed to the symmetric point discussed in ref. [25].

[7] Note that the remainder function in ref. [34] contains an additive constant $-\frac{23}{2}\zeta_4$, which we did not include in our definition (5.8).

Last but certainly not least is the question of the dual description of scattering amplitudes and form factors on the Coulomb branch. A proposal for an off-shell Wilson loop was put forward in ref. [70] starting from a higher-dimensional holonomy and dimensionally reducing it down to four-dimensions. However, while the one-loop expectation value for four sites was found to be in agreement with the amplitude of the W-bosons, starting from two loops the two 'observables' started to deviate. The reason for this fact remains obscure. The T-dual gauge theory was chosen to be the conformal $\mathcal{N}=4$ sYM. Had it rather be something else or one had to use a different variant of dimensional reduction? This question will have to be readdressed in the future.

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