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The Effect of Multiplicity of Stellar Encounters and the Diffusion Coefficients in a Locally Homogeneous Three-Dimensional Stellar Medium: Removing the Classical Divergence

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Abstract—Agekyan's λ-factor that allows for the effect of multiplicity of stellar encounters with large impact parameters has been used for the first time to directly calculate the diffusion coefficients in the phase space of a stellar system. Simple estimates show that the cumulative effect, i.e., the total contribution of distant encounters to the change in the velocity of a test star, given the multiplicity of stellar encounters, is finite, and the logarithmic divergence inherent in the classical description of diffusion is removed, as was shown previously by Kandrup using a different, more complex approach. In this case, the expressions for the diffusion coefficients, as in the classical description, contain the logarithm of the ratio of two independent quantities: the mean interparticle distance and the impact parameter of a close encounter. However, the physical meaning of this logarithmic factor changes radically: it reflects not the divergence but the presence of two characteristic length scales inherent in the stellar medium.

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INTRODUCTION

The present-day description of collisional processes in stellar systems dates back to the pioneering works in the first half of the 20th century (Charlier 1917; Jeans 1919; Spitzer 1924; Rosseland 1928; Smart 1938; Williamson and Chandrasekhar 1941; Chandrasekhar 1941a, 1941b, 1942). An exhaustive in-depth historical overview of the works on the kinetic theory of homogeneous systems with long-range interactions, including the stellar medium, can be found in Chavanis (2013). Chandrasekhar (1941a) was the first to use the the Holtsmark (1919) distribution for a random force in a homogeneous stellar medium and to show that the asymptotics of this distribution in the approximation of large forces completely coincides with the distribution of the force due to the interaction with the nearest neighbor (Hertz 1909). This very important circumstance underlies the collisional kinetics of stellar systems that considers the changes in stellar velocities in terms of the hypothesis of binary encounters. Indeed, large (in magnitude) random changes in stellar velocities arise during close encounters of a test star with field stars at a characteristic distance much smaller than the mean interparticle distance. The cumulative effect of a series of such successive encounters considered as an independent random process of velocity change allows one to calculate the diffusion coefficients in velocity space and to derive the expressions for the collisional term in the Boltzmann equation $\frac{\partial f(t,\vec{R},\vec{V})}{\partial t} = -\left(\frac{\partial f}{\partial t}\right)_{\rm coll}$ in the Fokker–Planck approximation. The consistent development of the theory of irregular forces in stellar dynamics in the 20th century was based precisely on these basic principles.

There are several methods for estimating the cumulative effect: from the deflection angle of the velocity vector of a test star (Williamson and Chandrasekhar 1941; Parenago 1954), from the rate of change of the parallel (dynamical friction) and normal (scattering or diffusion) velocity components of a test star (Chandrasekhar 1941a; King 2002; Binney and Tremaine 2008; etc.). All estimates of the rate of change of the velocity and kinetic energy usually lead to close values of the time scale for these processes, which is often identified with the collisional relaxation time. A characteristic feature of the expressions for the diffusion coefficients and the relaxation time is the

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presence of a logarithmic divergence at the upper limit of integration over the impact parameter within the concept of successive independent binary encounters. The logarithmic factor $\Lambda = \ln \frac{d_{\max}}{p_{90}}$, an analog of the so-called Coulomb logarithm encountered in plasma physics, enters into the expressions for the linear and quadratic diffusion coefficients $\langle \Delta V_{\|} \rangle$ and $\langle \Delta V_{\perp}^2 \rangle$. Here, $d_{\max}$ is the upper limit of the impact parameter, $p_{90} = \frac{G(m+m_f)}{V_0^2}$ is the impact parameter of a close encounter at which the relative velocity vector of the encountering stars is deflected through 90°, where G is the gravitational constant, m and m_f are the masses of the test and field stars, respectively, and V_0 is the magnitude of the relative stellar velocity.

The problem of the upper limit of integration over the impact parameter has been repeatedly discussed in the works on stellar dynamics. For example, Williamson and Chandrasekhar (1941), Parenago (1954), and Henon (1958) pointed out that within the concept of binary encounters the mean interparticle distance, $\bar{d} \approx 0.554\nu^{-1/3}$ (here, ν is the mean number density of the stellar medium), is a natural upper limit of the impact parameter, because all weaker encounters are actually multiple and incomplete ones. The changes in the relative velocity are described by the formulas of hyperbolic relative motion under the assumption of their completeness (see, e.g., Binney and Tremaine 2008, p. 155), which are inapplicable for distant encounters whose formal allowance leads to a significant overestimation of the cumulative effect. We share this viewpoint and note that the treatment of all encounters (including the distant ones) as binary ones actually incorporates not only the irregular forces but also to some extent the regular force component into the diffusion coefficients being calculated. Ambartsumyan (1938), Ogorodnikov (1958), Binney and Tremaine (2008), and other authors mentioned in the latter monograph, on the contrary, believe that $d_{\max}$ should be set equal to the characteristic size of the entire stellar system (the star cluster radius, the galactic disk thickness, the Jeans length, the size of a regular stellar orbit, the size of a characteristic inhomogeneity in the stellar system). Note, however, that the precise choice of a maximum impact parameter is not critical for practical purposes (calculating the diffusion coefficients, estimating the characteristic time scales) due to the comparatively weak logarithmic divergence that cannot change radically the dynamical estimates. Indeed, in the solar neighborhood of our Galaxy $\bar{d} \approx 1$ pc, $p_{90} \approx 1$–2 AU, and $\Lambda \approx \ln \frac{\bar{d}}{p_{90}} \sim 11$–12. Choosing $d_{\max} \sim H_z \approx 100$ pc (the effective thin-disk thickness), we will increase the Coulomb logarithm Λ to $\Lambda \sim 15$–16, i.e., only by 40–45%, which will not lead to a radical change in all our dynamical estimates. Nevertheless, the problem of optimally choosing an upper limit for the impact parameter has another aspect directly related to the physical justifications of the collisional kinetics of stellar systems. We are sure that a deeper understanding of these phenomena and attempts to describe them in a consistent way are of fundamental importance for stellar dynamics. It is this problem that is discussed in our work.

MULTIPLICITY OF STELLAR ENCOUNTERS

Agekyan (1959, 1962) proposed and implemented a probabilistic approach to the description of stellar encounters and derived an analytical expression for the probability of a stellar encounter $\Phi\left(V^2, h\right)$ as a result of which the velocity of the test star changes by a specified value for several special cases. Here, $h = \frac{\Delta V^2}{V^2}$, ΔV^2 is the change in the squared velocity (i.e., the kinetic energy) of the test star. A weak point of Agekyan's approach was the divergence of the expression for the probability for small change in velocity, $\Phi(V^2, h) \sim |h|^{-3}$, which, in particular, prevented the calculation of the average change in stellar energy. It is obvious that the multiplicity of stellar encounters, which provide small changes in velocity, is responsible for this divergence. To mitigate the divergence effect, Agekyan (1961) introduced a reduction factor that allows for the multiplicity of encounters (Agekyan's λ-factor). It is numerically equal to the ratio of the magnitude of the total random force $\left|\delta\vec{F}\right|$ acting on the test star from all stars in a thin spherical layer (p, $p + dp$) surrounding the test star to the arithmetic sum of the magnitudes of the forces $\sum\left|\vec{F}_i\right|$ produced by the same stars, i.e.,

$$\lambda\left(p\right) = \frac{\left|\delta\vec{F}\right|}{\sum\left|\vec{F}_i\right|} < 1. \qquad (1)$$

The physical meaning of the λ-factor is that the total random force is equal to the geometric sum of the force vectors from all stars, while the classical calculations of the cumulative effect actually realize a simple arithmetic summation of the effects from individual encounters that are deemed independent. Obviously, the true role of each field star (the contribution to the random force vector $\left|\delta\vec{F}\right|$) is, on average, smaller than follows from the calculations of the result of its binary encounter with the test star due to the gravitational leveling of the contributions from distant stars. As the impact parameter increases, the numerator in Eq. (1) decreases, while the denominator grows. The rapid decrease in the λ-factor with distance is directly related to the fact that the angular distribution of field stars at large distances becomes increasingly uniform, and the contributions of individual stars to the random force effectively cancel each other out. At the same time, the random force acting on the test star must be determined only by the distribution of nearest neighbors, where the polarization of the angular distribution is larger, and the leveling effect is comparatively small. Therefore, one might expect the upper limit of integration over the impact parameter to be no greater than several mean interparticle distances $\bar{d}$.

These problems were subsequently considered by Kandrup (1980), who reached the qualitative conclusion that the forces acting from distant stars must effectively cancel each other out. A quantitative description of the field of irregular forces in a locally homogeneous stellar medium is given in his next fundamental paper (Kandrup 1981). He was the first to note that the integration of the diffusion coefficients in the collisional Fokker–Planck term over the impact parameter should not lead to any divergence at the upper limit. A similar conclusion was reached by Petrovskaya (1992), who investigated a thin gravitating stellar layer.

In this paper for the first time we calculate the diffusion coefficients by taking into account Agekyan's reduction factor (1) that compensates the overestimation of the contribution from distant encounters to the change in velocity for a three-dimensional Poissonian medium. To calculate the reduction factor $\lambda(p)$, Agekyan (1961) applied a technique leading to the Holtsmark (1919) distribution and obtained the following rigorous expression for a homogeneous infinite medium with a mean number density ν:

$$\lambda\,(p) = \frac{4}{\pi}\int\limits_0^\infty \frac{x-\sin x}{x^3}\exp\left(-a\frac{4\pi}{3}\nu p^3 x^{3/2}\right)dx, \quad (2)$$

where $a = \frac{2}{5}\sqrt{2\pi} \approx 1.00265$. Given that $\frac{4\pi}{3}\nu p^3 \equiv N\,(p)$ is the average number of field stars within the sphere of radius p, where p is the impact parameter of the encounter under consideration, let us rewrite (2) in an equivalent form by assuming the λ-factor to be a function of $N = N(p)$:

$$\lambda\,(N) = \int\limits_0^\infty \frac{x-\sin x}{x^3}\,\exp\left(-aNx^{3/2}\right)dx. \quad (3)$$

Thus, Agekyan's λ-factor is completely determined by the average number of stars within the sphere whose radius is equal to the impact parameter of the encounter under consideration. This will be used in the text of the paper. The reduction factor $\lambda(p)$ at large N shows a well-known asymptotic behavior, $\lambda(N) \sim N^{-2/3} \sim p^{-2}$ (Agekyan 1961), and rapidly decreases with increasing impact parameter. Figure 1 shows the dependence of Agekyan's λ-factor on the impact parameter expressed in units of the mean interparticle distance $p' = p/\bar{d}$. The encounter multiplicity effect is seen to overestimate the random force by more than an order of magnitude even at two mean interparticle distances. This fact qualitatively justifies the intuitive conclusion of Williamson and Chandrasekhar (1941) that in a 3D Poisson model of the stellar medium the nearest neighbors to the test star make a major contribution to the random gravitational force.

Strictly speaking, it will be logically correct to use Agekyan's λ-factor to take into account the contribution of stellar encounters directly in the integration over the impact parameter in an ordinary scheme of calculations of the diffusion coefficients in the Fokker–Planck approximation (see, e.g., Eqs. (L.1)–(L.26) in the monograph by Binney and Tremaine (2008)) or when deducing the probability of an encounter with a given change in velocity describing a Markov process of velocity changes with the Kolmogorov–Feller collisional integral term (Agekyan 1959; Petrovskaya 1969a, 1969b). Note that Agekyan obtained an explicit form of the λ-factor two years after the publication of his first fundamental paper, where he proposed the probabilistic approach to describing the random process of change in the velocity of a test particle (Agekyan 1959). Based on the great complexity of the method for deriving the probability of an encounter with a given change in velocity used by Agekyan (1959, 1962), we can assume that it is unlikely that he would be able to derive an analytical expression for this probability even for the simplest cases if the λ-factor were immediately built into the algorithm of integration over the impact parameter. Therefore, Agekyan (1961, 1962) took a palliative decision and introduced a correction factor to the previously found probability. The probability corrected in this way is

$$\tilde{\Phi}(V^2, h) = \lambda(\bar{p})\Phi(V^2, h), \quad (4)$$

where $\bar{p}$ is some effective mean impact parameter at which the encounter with a field star leads to a relative change in the squared velocity, on average, by h. In this case, it is possible to "soften" the divergence of the expression for the probability at large impact parameters (or small relative changes in the squared velocity) to $\tilde{\Phi}(V^2, h) \sim |h|^{-1}$ (Agekyan 1961), while the probability correction method itself is a corollary of the integral mean-value theorem (Fikhtengolts 1969).

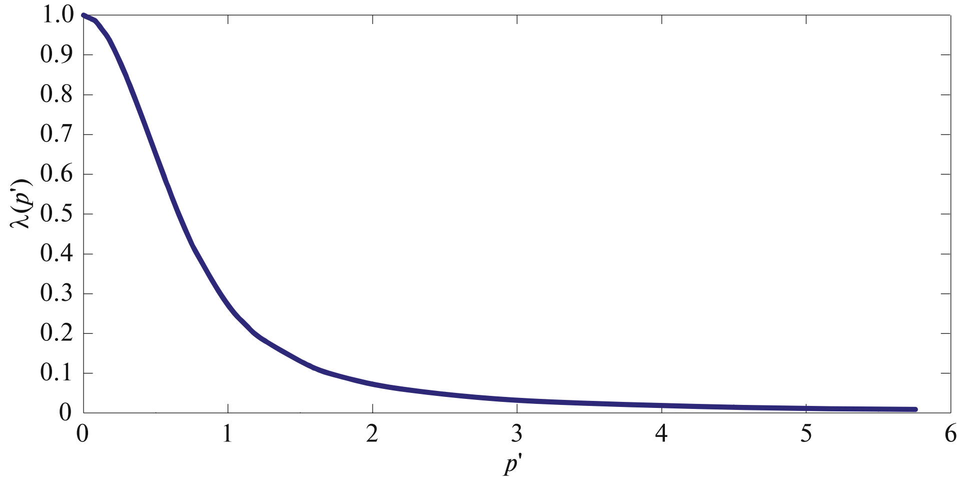

Fig. 1. Dependence of Agekyan's λ-factor on impact parameter $p\prime$ expressed in units of the mean interparticle distance $\bar{d}$.

The probabilistic approach turned out to be highly efficient in problems of analytical stellar dynamics but, in our opinion, not popular enough due to the extreme complexity of calculations. For example, Petrovskaya (1969a, 1969b) used it to describe the change in the velocity of a test star in an irregular force field in terms of a purely discontinuous random process. Indeed, the corrected probability of an encounter (4) can be directly substituted into the expression for the collisional term in the most general form of the balance equation (Eq. (7.62)) in the monograph by Binney and Tremaine (2008). As a result, it will take the form of a Kolmogorov—Feller integro-differential equation for the phase-space density. In their series of papers Kaliberda and Petrovskaya (1970, 1971, 1972) and Kaliberda (1971, 1972) derived the equilibrium solutions of the Kolmogorov—Feller equation for the local velocity distribution of stars of different masses using numerical methods. An indubitable advantage of the description of collisional kinetics in terms of a purely discontinuous random process compared to the classical description of diffusion in velocity space is the possibility of a direct calculation of not only the mass loss rate but also the energy losses, which, obviously, accelerates the dynamical evolution of the stellar system and reduces its lifetime.

RIGOROUS ALLOWANCE FOR THE MULTIPLICITY OF STELLAR ENCOUNTERS IN A THREE-DIMENSIONAL MEDIUM IN THE DIFFUSION APPROXIMATION

Let us consider the problem of calculating the diffusion coefficients in a homogeneous infinite three-dimensional medium by taking into account the multiplicity of stellar encounters and using the fundamental results of Agekyan (1961). This problem seems quite solvable in terms of the classical approach, in contrast to the difficulties with the probabilistic approach that we noted above. In contrast to the approach of Kandrup (1981), we will perform direct integration of the diffusion coefficients over the impact parameter to understand whether the logarithmic divergence is retained on large scales.

When deriving the modified expressions for the diffusion coefficients, we will rely on the classical approach described in the monograph by Binney and Tremaine (2008, Appendix L.6). The initial expressions for the diffusion tensor components in velocity space for a test star averaged over the orientation angle of the relative orbit of the encountering stars are $(k, l = 1, 2, 3)$

$$\langle \Delta V_k \rangle = -\Delta V_{\parallel} \left(\overrightarrow{e_k} \cdot \overrightarrow{e_1'} \right), \tag{5}$$

$$\begin{aligned} \langle \Delta V_k \Delta V \rangle_l &= \left(\Delta V_{\parallel}\right)^2 \left(\overrightarrow{e_k} \cdot \overrightarrow{e_1'} \right) \left(\overrightarrow{e_l} \cdot \overrightarrow{e_1'} \right) \\ &+ \frac{1}{2} \left(\Delta V_{\perp}\right)^2 \left[\left(\overrightarrow{e_k} \cdot \overrightarrow{e_2'} \right) \left(\overrightarrow{e_l} \cdot \overrightarrow{e_2'} \right) \right. \\ &\left. + \left(\overrightarrow{e_k} \cdot \overrightarrow{e_3'} \right) \left(\overrightarrow{e_l} \cdot \overrightarrow{e_3'} \right) \right], \end{aligned} \tag{6}$$

where $(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$ are the unit vectors of the laboratory coordinate system, and $\left(\overrightarrow{e_1'}, \overrightarrow{e_2'}, \overrightarrow{e_3'}\right)$ are the unit vectors of the coordinate system associated with the center of mass of the encountering stars, with the unit vector $\overrightarrow{e_1'}$ being directed along the vector of their initial relative velocity $\overrightarrow{V_0} = \left(\overrightarrow{V''}\right) - \overrightarrow{V_f}$ (see Fig. L.1 in the monograph of Binney and Tremaine (2008)). The changes in the longitudinal and transverse velocity components appearing in Eqs. (5) and (6) for the diffusion coefficients transformed to the laboratory frame of reference will be, respectively (ibid., Fig. L.7)

$$\Delta V_{\parallel} = \frac{2Gm_f p_{90}}{V_0\left(p^2 + p_{90}^2\right)}, \quad \Delta V_{\perp} = \frac{2Gm_f p}{V_0\left(p^2 + p_{90}^2\right)}, \quad (7)$$

where m_f, V_0, and p are the mass of the field star, the magnitude of the relative velocity, and the impact parameter, respectively.

The change in the velocity component $\Delta V_{\parallel}$ is calculated in a unit time interval and, consequently, can be treated as an acceleration of the test star (to be more precise, a deceleration due to dynamical friction) in the stochastic field of irregular forces. The logic of reasoning suggests that, on the one hand, the velocity change $\Delta V_{\parallel}$ should be integrated over the impact parameter in this case with a weight $\lambda(p)$ equal to the reduction factor for the stochastic force acting on the test star. On the other hand, the coefficients $\left(\Delta V_{\parallel}\right)^2$ and $\left(\Delta V_{\perp}\right)^2$ may be considered as the changes in the components of the kinetic energy tensor per unit time, which within our concept are proportional to the work of the random force; this means that they should be integrated over the impact parameter with the same weight $\lambda(p)$.

As in the classical case, we take into account the fact that the test star experiences $d\eta(p) = 2\pi\nu_f V_0 p dp$ encounters with field stars with a relative velocity V_0 and impact parameters in the interval $(p, p+dp)$ per unit time. Recall that ν_f is the number density of field stars with the relative velocity $\overrightarrow{V_0}$, i.e., $\nu_f = f\left(\overrightarrow{V_f}\right) d\overrightarrow{V_f}$ and $\overrightarrow{V_f} = \overrightarrow{V} - \overrightarrow{V_0}$, where $\overrightarrow{V_f}$ is the velocity of the field star and $f\left(\overrightarrow{V_f}\right)$ is the velocity distribution function of field stars. Here, we will restrict ourselves only to the integration over the impact parameter, because the subsequent integration over the velocity distribution of field stars is performed in the same way as in the classical works on stellar dynamics (see, e.g., Appendix L in the monograph of Binney and Tremaine (2008)), leading to Rosenbluth potentials (Rosenbluth et al. 1957).

Agekyan's λ-factor cannot be expressed in terms of elementary functions and, hence, we use its simple piecewise continuous approximation. First, for convenience (as will be clear from the subsequent discussion), we will consider the λ-factor to be a function of $\tilde{\lambda}(n)$, where $n = (N/N_0)^{2/3}$ and $N_0 \approx 0.7122$ is the average number of stars (expectation) within the sphere with a radius equal to the mean interparticle distance $\bar{d}$. Consequently, we naturally introduce $\bar{d}$ as a scale parameter of the stellar medium. Using numerical methods, we found the following approximation of the λ-factor with an accuracy of $\sim$2–3%:

$$\tilde{\lambda}(n) = \begin{cases} a\exp[-bn^c] + d, & n \le 1, \\ en^{-1}, & n > 1, \end{cases} \quad (8)$$

where $a \approx 0.863 \pm 0.001$, $b \approx 2.281 \pm 0.002$, $c \approx 0.924 \pm 0.0005$, $d \approx 0.141 \pm 0.002$, and $e \approx 0.235 \pm 0.001$ (at 95% confidence). This accuracy is quite sufficient for estimating the integrals of the velocity changes (7).

The current values of the impact parameter and the average number of stars in the sphere of the corresponding radius are related by the following obvious relation:

$$p^2 = \bar{d}^2\,(N/N_0)^{2/3} = \bar{d}^2 n. \quad (9)$$

The next step in calculating the diffusion coefficients is to integrate the velocity changes $\Delta V_{\parallel}$, $\left(\Delta V_{\parallel}\right)^2$, and $\left(\Delta V_{\perp}\right)^2$ over the impact parameter with a weight equal to $d\eta(p) = 2\pi\nu_f V_0 p dp$:

$$DV_{\parallel} = \pi\nu_f V_0 \int_0^{\infty} \Delta V_{\parallel} \tilde{\lambda}(n)\, d\left(p^2\right), \quad (10)$$

$$DV_{\parallel}^2 = \pi\nu_f V_0 \int_0^{\infty} \left(\Delta V_{\parallel}\right)^2 \tilde{\lambda}(n)\, d\left(p^2\right), \quad (11)$$

$$DV_{\perp}^2 = \pi\nu_f V_0 \int_0^{\infty} \left(\Delta V_{\perp}\right)^2 \tilde{\lambda}(n)\, d\left(p^2\right). \quad (12)$$

It is obvious that $DV_{\parallel}^2 \ll DV_{\perp}^2$, because the integral (11) converges and the noted inequality holds already in the classical approximation. The convergence will be even more evident in our case after multiplying the integrand by the rapidly decreasing function $\tilde{\lambda}(n)$. For this reason, below we may also neglect the contribution of (11) to the diffusion coefficient (6).

Next, we will use Eq. (9) and pass from the integration over the impact parameter to the integration over the variable n by transforming the linear diffusion (dynamical friction) coefficient (10) to the form

$$DV_{\parallel} = \frac{2\pi G^2 m_f (m + m_f)\nu_f}{V_0^2} \quad (13)$$

$$\times \int_0^{\infty} \frac{\tilde{\lambda}(n)\, d\left(p^2\right)}{\left(p^2 + p_{90}^2\right)} = \frac{2\pi G^2 m_f (m + m_f)\nu_f}{V_0^2} K^2$$

$$\times \int_0^{\infty} \frac{\tilde{\lambda}(n)\, dn}{1 + K^2 n} = \frac{2\pi G^2 m_f (m + m_f)\nu_f}{V_0^2} I_1(K),$$

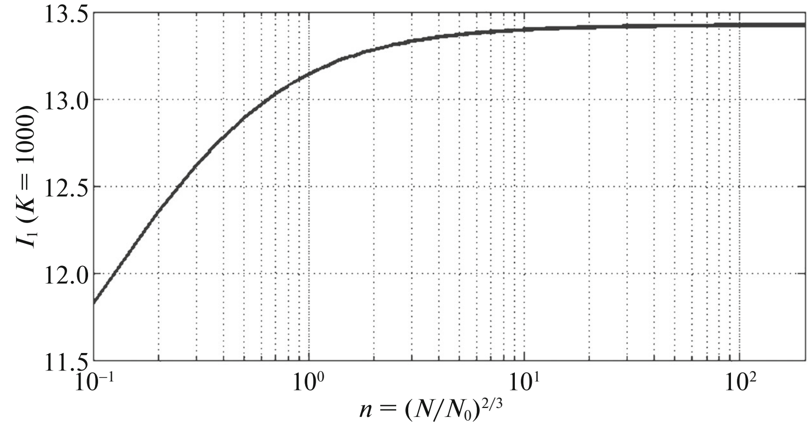

Fig. 2. Behavior of the integral I_1 ($K = 1000$) in Eq. (13) as a function of the upper limit of integration $n_{\max}$.

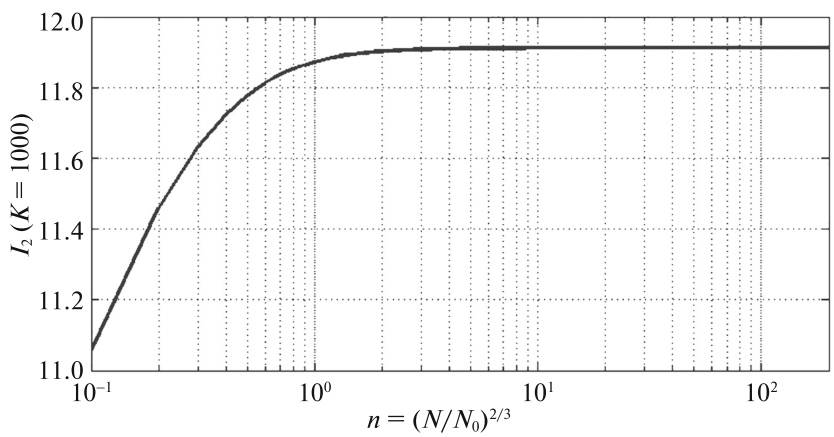

Fig. 3. Behavior of the integral I_2 ($K = 1000$) in Eq. (14) as a function of the upper limit of integration $n_{\max}$.

where $K = \bar{d}/p_{90}$ is the ratio of two characteristic length scales of the stellar medium. We similarly derive the expression for the quadratic diffusion coefficient:

$$DV_{\perp}^2 = \frac{4\pi G^2 m_f^2 \nu_f}{V_0} \int_0^{\infty} \frac{\tilde{\lambda}(n)\, p^2 d\left(p^2\right)}{\left(p^2 + p_{90}^2\right)} \tag{14}$$

$$= \frac{4\pi G^2 m_f^2 \nu_f}{V_0} K^4 \int_0^{\infty} \frac{\tilde{\lambda}(n)\, n dn}{(1 + K^2 n)^2}$$

$$= \frac{4\pi G^2 m_f^2 \nu_f}{V_0} I_2(K),$$

where we introduce the dimensionless functions

$$I_1(K) = K^2 \int_0^{\infty} \frac{\tilde{\lambda}(n)\, dn}{1 + K^2 n}, \tag{15}$$

$$I_2(K) = K^4 \int_0^{\infty} \frac{\tilde{\lambda}(n)\, n dn}{(1 + K^2 n)^2},$$

which are dependent on the ratio of the scale factors K and enter into Eqs. (13) and (14) for the diffusion coefficients.

We calculated the integrals (15) by numerical integration for a wide range of ratios K ($1 < K < 10^5$). Note that the upper boundary of the parameter K corresponds to a low number density of stars, $\sim$0.1 pc^{-3}, typical for the solar neighborhood. Figure 2 shows the behavior of the integral I_1 with increasing upper limit of integration for $K = \bar{d}/p_{90} = 1000$. We see that the integral reaches a plateau already at relatively small values of the upper limit $n_{\max}$, demonstrating the complete absence of a logarithmic divergence. As follows from (9), $n_{\max} = \left(p_{\max}/\bar{d}\right)^2 \equiv \left(d_{\max}/\bar{d}\right)^2$; hence the integral is close to its limiting value at an impact parameter of $\sim$2–3 mean interparticle distances, where distant encounters are "shielded" almost completely.

Figure 3 shows the behavior of the integral I_2 as a function of the upper limit of integration also for $K = \bar{d}/p_{90} = 1000$. We see that this integral converges

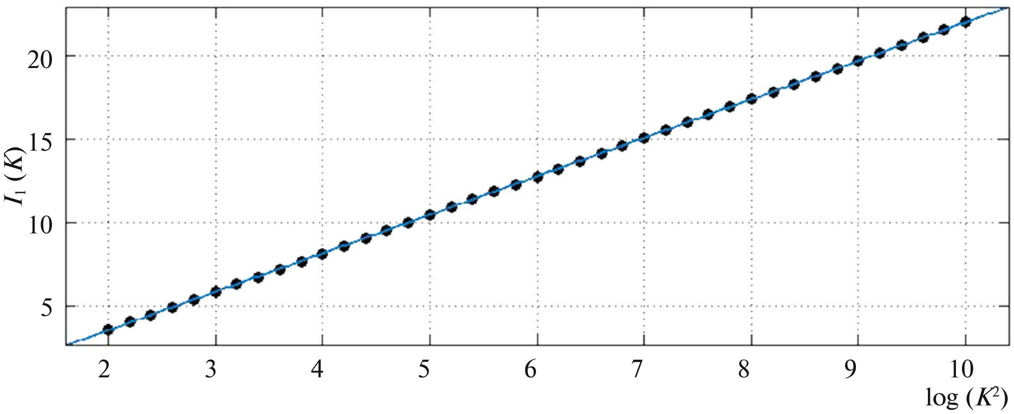

Fig. 4. Dependence of the integral I_1 in Eq. (13) on parameter K in the range $10 < K < 10^5$. The dots indicate the results of our calculations with a constant step in $\log(K^2)$; the solid line is a linear approximation of the integral as a function of $\log(K^2)$ (the 95% confidence intervals are smaller than the size of the symbols).

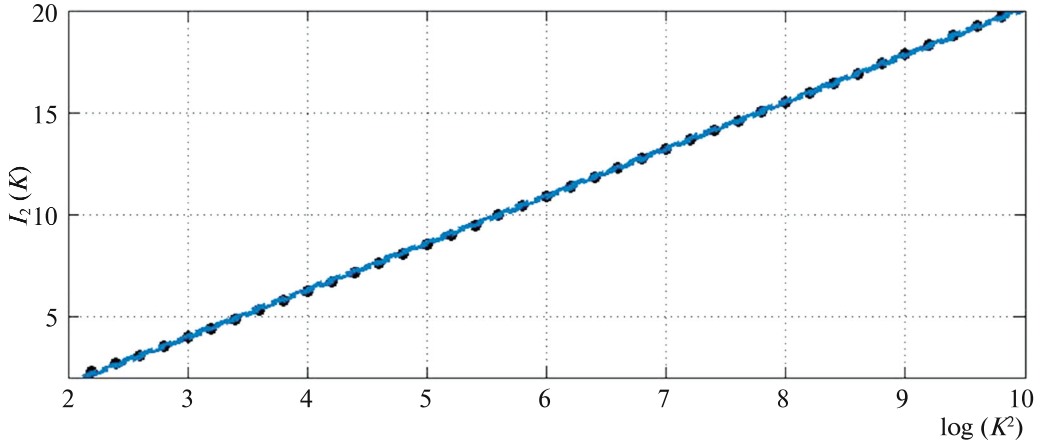

Fig. 5. Dependence of the integral I_2 in Eq. (14) on parameter K in the range $10 < K < 10^5$. The designations are the same as those in Fig. 4.

even faster, and an efficient "shielding" of distant encounters begins already from 1–2 mean interparticle distances. This is because Agekyan's λ-factor decreases rapidly with increasing impact parameter.

Figures 4 and 5 show the dependence of the integrals I_1 and I_2 on the ratio of the scale factors K.

The parameters of the linear dependences of I_1 and I_2 on $\log(K^2)$ for $K > 10$ can be estimated from the results of our calculations shown in Figs. 4 and 5:

$$I_1\,(K) \approx (2.306 \pm 0.010)\log\left(K^2\right) - (1.070 \pm 0.030) \approx 2\ln\left(K/1.7\right), \quad (16)$$

$$I_2\,(K) \approx (2.302 \pm 0.040)\log\left(K^2\right) - (2.224 \pm 0.020) \approx 2\ln\left(K/3.0\right), \quad (17)$$

where ln stands for the natural logarithm. Substituting Eqs. (16) and (17) for the integrals into Eqs. (13) and (14) for the diffusion coefficients, respectively, we will obtain the final expressions for the linear and quadratic diffusion coefficients will allowance made for the gravitational "shielding" of distant encounters:

$$DV_{\parallel} \approx \frac{4\pi G^2 m_f\,(m+m_f)\,\nu_f}{V_0^2}\ln\left(\bar{d}/1.7p_{90}\right), \quad (18)$$

$$DV_{\perp}^2 \approx \frac{8\pi G^2 m_f^2 \nu_f}{V_0}\ln\left(\bar{d}/3.0p_{90}\right). \quad (19)$$

DISCUSSION OF RESULTS

Let us compare our calculated diffusion coefficients with the results of classical calculations with the intuitive "cutoff" of distant encounters at the mean interparticle distance (see, e.g., Williamson and Chandrasekhar 1941):

$$DV_{\parallel} \approx \frac{4\pi G^2 m_f\,(m+m_f)\,\nu_f}{V_0^2}\ln\left(\bar{d}/p_{90}\right), \quad (20)$$

$$DV_\perp^2 \approx \frac{8\pi G^2 m_f^2 \nu_f}{V_0} \ln\left(\bar{d}/\sqrt{e}p_{90}\right). \qquad (21)$$

Expressions (18) and (19) show that in our approximation the ratios under the logarithm are smaller than the corresponding terms in the classical expressions (20) and (21) by a factor of 1.7–1.8. From a practical point of view, this small difference of the logarithms (corresponding to about 0.55, i.e., ∼5–10% of the Coulomb logarithm Λ) plays absolutely no role in the calculations of the diffusion coefficients by a particular method and the corresponding characteristic time scales. However, from a conceptual point of view, this means that when calculating the cumulative effect within the concept of binary encounters, the effective upper boundary of the impact parameter should hardly exceed the mean interparticle distance. More distant stars more likely contribute to the regular force or the density fluctuations, which should be taken into account either within the framework of collective interactions or using the methods of fractal analysis.

Thus, our results on both logical and quantitative levels support the viewpoint of those researchers that previously restricted the upper limit of integration over the impact parameter by the mean interparticle distance from entirely reasonable considerations. Based on (18) and (19), we can say that encounters with impact parameters $p \lesssim 0.6\bar{d}$ may be considered as truly independent and complete in terms of the classical approach, while encounters with $p \gtrsim (2-3)\,\bar{d}$ actually make no contribution to the irregular force in a homogeneous Poissonian system.

Finally, even more importantly, Eqs. (18) and (19) contain a logarithmic factor, as in the classical concept of the cumulative effect. However, its physical meaning in our approach differs fundamentally from the classical case. For example, we showed that allowance for the multiplicity of distant encounters and the gravitational "shielding" removes the divergence at the upper limit. In our case, the logarithmic factor appears quite naturally and reflects the presence of two independent length scales in the stellar medium: the mean interparticle distance $\bar{d} \approx 0.554\nu^{-1/3}$, which is determined only by the mean number density of stars, and the close-encounter parameter $p_{90} = \frac{G(m+m_f)}{V_0^2}$, which reflects the dynamical properties of the stellar field (dependent on the masses and characteristic velocities of stars). It is important to note that these two fundamental parameters will be related to each other only under virial equilibrium conditions.

Virtually the same conclusions were previously reached by Kandrup (1981), who analyzed the kinetic processes in a locally homogeneous stellar medium (i.e., homogeneous on scales of the order of several mean interparticle distances). Using a distribution of random forces similar to the Holtsmark distribution, he rigorously derived expressions for the diffusion coefficients (Eqs. (139) and (140) in his paper) coincident with the classical expressions for a homogeneous stellar medium with the mean interparticle distance as the upper limit of integration. Let us quote the concluding phrase from his paper (p. 1059, Kandrup 1981): "This equation (139), (140) is precisely the standard Fokker–Plank equation of conventional stellar dynamics, differing only in that here the logarithmic factor is not a divergence, but instead the ratio of two well-defined lengths. The basic conclusion of this stochastic analysis, therefore, is that the effects of nearby particles are adequately described in a binary encounter approximation and that because of statistical cancellations very distant particles contribute negligibly to the effects of fluctuations."

It can be seen from this text that our conclusions about the efficient gravitational "shielding" of encounters with impact parameters exceeding the mean interparticle distance are qualitatively and quantitatively very similar to the conclusions of Kandrup (1981) but were reached in a much simpler and transparent way. Consequently, an artificial cutoff of the impact parameter when calculating the cumulative effect seems superfluous.

We calculated Agekyan's λ-factor based on the Holtsmark distribution for an infinite homogeneous stellar medium. In real stellar systems the characteristic size of spatial irregularities (density fluctuations) exceeds appreciably the mean interparticle distance. Consequently, our proposed method of allowance for the irregular forces with an effective cutoff on scales comparable to the mean interparticle distance is also well suited for the description of inhomogeneous systems, as was also pointed out, in particular, by Kandrup (1981). Obviously, the influence of spatial irregularities on the stellar kinetics can manifest itself through collective effects or effects associated with the fractal structure of the medium (Chumak and Rastorguev 2015). Vlad (1994), Chavanis (2009), and Chumak and Rastorguev (2015, 2016) showed that the distribution of the random force in a fractal medium could be described by a complete analog of the Holtsmark distribution, where the mean number density of stars is replaced by the conditional density calculated based on the fractal dimension of the system. On this basis, Chumak and Rastorguev (2017) performed detailed calculations of the influence of the random force on the stellar kinetics in a fractal medium. We also believe that fractal media may be deemed locally homogeneous within several "intercluster" distances.

Analogies are often drawn between the descriptions of a plasma and a stellar system ("graviplasma"). We would like to note in this connection that, despite a certain similarity of the descriptive apparatus for these media, there are serious differences between them. For example, the mean interparticle distance in a "graviplasma" must serve as an analog of the Debye screening length in a plasma (the size of the region beyond which the plasma may be deemed electrically neutral), as is shown by our results and Kandrup (1981).

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