ISSN 0027-1349, Moscow University Physics Bulletin, 2024, Vol. 79, No. 5, pp. 627–633.

ASTRONOMY, ASTROPHYSICS, AND COSMOLOGY

Precession of Stellar Rings in the Center of Galaxy

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Received June 5, 2024; revised June 27, 2024; accepted July 2, 2024

Abstract—The dynamics of two interacting concentric wide star rings located around a supermassive black hole in the central parsec of the Galaxy is investigated. The most massive of them, the ring with the retrograde motion of stars (known in the literature as a “clockwise” disk) is modeled as R-disk with a small cutout in the center. Another ring with a direct motion of stars (“counter-clockwise” disk) is represented by a thin circular ring with an inclination $\alpha \approx 62^\circ$ to the plane of the R-disk. The masses of these rings M_1, M_2 ($M_1/M_2 \approx 60$), their geometric parameters and spatial orientation are known from observations. The mutual gravitational energy $W_{\rm mut}$ and the angular moment of force $\mathbf{M}$ between the rings have been found, and graphs of these quantities depending on the angle of inclination have been constructed. The angular momenta of the rings $L^{(1)}$ and $L^{(2)}$, whose ratio $L^{(1)}/L^{(2)} \approx 23.4$ have also been calculated. For the system of rings, the Laplace plane and the angles of its orientation are determined. It has been established that the mutual perturbation of the rotating rings leads to precession of the nodes with a period of $T_\Omega \approx 3.53 \times 10^5$ years. The lines of the nodes of both rings in the Laplace plane move with the same angular velocity but in opposite directions. This explains the large angle of divergence of the lines of nodes known from observations.

Keywords: disk models of stellar systems, mutual gravitational energy and moment of forces of rings, angular momentum, Laplace plane, nodal precession

DOI: 10.3103/S0027134924701911

INTRODUCTION

The study of the Galaxy showed that in the central parsec, in addition to the SMBH and the compact nuclear star cluster, there are also two concentric ring formations—disk 1 with the retrograde motion of stars, and disk 2 with the prograde motion of stars, which received their name from the direction of motion of stars observed in them [1–3]. Numerical modelling [4] confirmed the possibility of the existence of a second stellar disk. The interaction of these disks was studied in the articles [5, 6]. In work [7], the apsidal and nodal precession of the orbits of stars in disk 1 under the influence of a supermassive black hole and a nuclear star cluster were studied. In [7], the influence on the precession of orbits from the forces of disk 1 itself, modeled by the R-ring, was also taken into account. In [8], the tidal evolution of orbits in the galactic casp was studied.

According to the laws of celestial mechanics [9, 10], the mutual disturbance of rotating rings should lead to the precession of their planes and nodes. But much remains insufficiently studied in the dynamics of these ring structures. In particular, the secular precession of these stellar disks under the influence of their mutual attraction is now of great interest. Overall, this dynamic problem is very difficult. In [11], a simplified dynamic model was studied, where both disks were represented by weighted-mean narrow circular rings. The rings intersect in diameter, the angle of inclination between them, according to observations, is equal to $\alpha \approx 62^\circ$ In this model, the precession time of the disk nodes in the Laplace plane turned out to be equal $T_{\rm node} \approx 7 \times 10^7$ years. It is interesting to note that this time is almost an order of magnitude less than the time $T_{\rm aps} \approx 5 \times 10^8$ years of the apsidal precession of stars under the influence of the central black hole.

In this paper, we fill some gaps in the study of the dynamics of two nuclear stellar disks. In Section 1,

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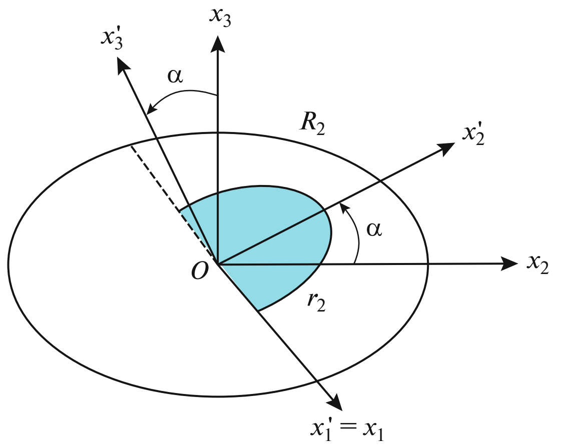

Fig. 1. Scheme of two circular rings intersecting along the axis Ox_1 at an angle α. The plane of the thin ring is shown in blue in the coordinates Ox_1x_2', and the plane of the wide R-ring is shown in white in the coordinates Ox_1x_2. The scale of the outer radii of the rings is not observed.

a model of a wide R-ring is introduced to describe the massive disk 1, which is closer to reality than the narrow ring model in [11]. Another ring star system (disk 2) is represented by a weighted average narrow circular ring with an inclination $\alpha \approx 62°$ to the disk plane. In Section 2 the mutual gravitational energy of the rings W_{mut} is calculated here in analytical and numerical versions. In Section 3, the moment of forces between the two rings is calculated by differentiating W_{mut} with respect to the angle of inclination. In Section 4, the angular momenta for each disk are found and the Laplace plane for the system as a whole is determined. This made it possible in Section 5 to calculate the period of precession of nodes in this plane for the specified disk system.

1. STATEMENT OF THE PROBLEM AND POTENTIAL OF THE R-DISK

Let us consider a wide inhomogeneous circular ring with inner R_1 and outer R_2 radii (Fig. 1). At the center of this ring is a supermassive black hole. It is known from observations that many stars move around SMBHs in orbits whose apsidal lines are uniformly distributed over the azimuthal angle. As shown in [12, 13], a model for such a disk can be an R-disk with a density distribution

$$\sigma\left(r'\right) = \frac{M_1}{\pi^2\left(R_1 + R_2\right)\sqrt{\left(R_2 - r'\right)\left(r' - R_1\right)}}. \quad (1)$$

A direct check using the formula

$$M_1 = 2\pi \int\limits_{R_1}^{R_2} r'\sigma\left(r'\right) dr', \quad (2)$$

shows that the R-ring does indeed have a mass M_1.

The potential of the R-ring at the point (r, x_3) is equal to [12, 13]:

$$\phi\left(r, x_3\right) = \frac{4GM_1}{\pi^2\left(R_1 + R_2\right)}$$

$$\times \int\limits_{R_1}^{R_2} \frac{r'dr'}{\sqrt{\left(R_2 - r'\right)\left(r' - R_1\right)}} \frac{K\left(k\right)}{\sqrt{\left(r + r'\right)^2 + x_3^2}}, \quad (3)$$

where $K\left(k\right)$ is the complete elliptic integral of the first kind with modulus

$$k = \sqrt{\frac{4rr'}{\left(r + r'\right)^2 + x_3^2}}. \quad (4)$$

This wide R-ring interacts with a narrow circular ring of radius r_2. These rings are concentric, the angle of inclination between their planes is $0 \leq \alpha \leq \pi/2$. In the Cartesian coordinate system Ox_1x_2, the wide ring is in the plane Ox_1x_2', and the thin ring is in the plane Ox_1x_2', with the axis Ox_2' inclined to the axis Ox_2 at an angle α. Then the coordinates of the test point on the thin ring can be expressed through the coordinates of the main coordinate system $Ox_1x_2x_3$:

$$x_1 = x' = r_2 \cos\theta,$$
$$x_2 = x_2' \cos\alpha = r_2 \sin\theta \cos\alpha,$$
$$x_3 = x_2' \sin\alpha = r_2 \sin\theta \sin\alpha, \quad (5)$$

where the coordinate angle $0 \leq \theta \leq 2\pi$ is measured along the ring from the axis Ox_1. Then, as is easy to see,

$$r = \sqrt{x_1^2 + x_2^2} = r_2\sqrt{n},$$
$$(r' + r)^2 + x_3^2 = r'^2 + r_2^2 + 2r'r_2\sqrt{n}, \quad (6)$$

where

$$n = \sin^2\theta\cos^2\alpha + \cos^2\theta. \quad (7)$$

Introducing an auxiliary quantity

$$m = \sqrt{\frac{r'^2 + r_2^2}{r'r_2} + 2\sqrt{n}}, \quad (8)$$

we write the potential of the wide ring (3) at the points of the narrow one in the form:

$$\phi\left(r, x_3\right) = \frac{4GM_1}{\pi^2\left(R_1 + R_2\right)r_2}$$

$$\times \int\limits_{R_1}^{R_2} \frac{\sqrt{r'}dr'}{\sqrt{\left(R_2 - r'\right)\left(r' - R_1\right)}} \frac{K\left(k\right)}{m}, \quad (9)$$

and the modulus of the elliptic integral of the first kind will be equal to

$$k = \frac{2n^{\frac{1}{4}}}{m}. \quad (10)$$

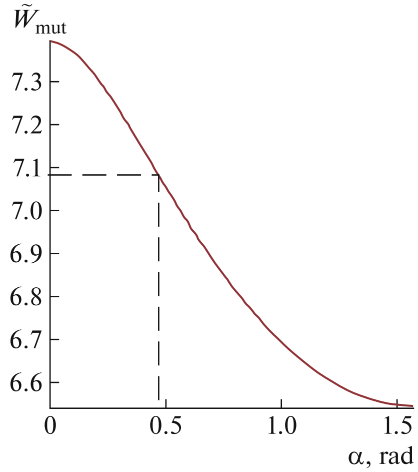

Fig. 2. Dependence of the normalized mutual energy $\tilde{W}_{\rm mut} = W_{\rm mut}/\left(-\frac{8GM_1M_2}{\pi^2\sqrt{a_1 r_2}}\right)$ for the system of the R-ring and the narrow ring on the angle of inclination α between them. The inflection point on the graph is shown by dashes. The parameters of the R-ring are taken from observations of disk 1: $R_1 = 1'' = 0.04$ pc, $R_1 = 10'' = 0.4$ pc. The narrow ring is a model of disk 2 with a radius $r_2 = 12a_1 = 0.48$ pc.

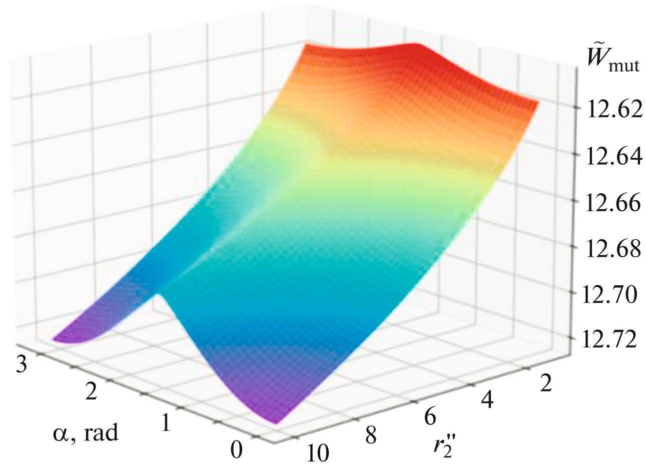

Fig. 3. Three-dimensional dependence of the normalized mutual energy $\tilde{W}_{\rm mut} = W_{\rm mut}/\left(-\frac{GM_1M_2}{\pi^2(R_1+R_2)}\right)$ between the R-ring and the narrow ring on the tilt angle α (in rad) and the radius r_2 (in arcseconds).

2. MUTUAL GRAVITATIONAL ENERGY

To find the mutual gravitational energy of a system of wide and narrow rings, we multiply potential (9) by the mass element of the narrow ring $\mu_2 r_2 d\theta$ and integrate over the coordinate $0 \le \theta \le 2\pi$. Here, $\mu_2 = \frac{M_2}{2\pi r_2}$ is the one-dimensional density on the second ring with mass M_2. After integration, we obtain an expression for the mutual gravitational energy in the form of a double integral:

$$W_{\rm mut} = -\frac{16GM_1M_2}{\pi^2(R_1+R_2)\sqrt{r_2}}$$

$$\times \int_0^{\frac{\pi}{2}} d\theta \int_{R_1}^{R_2} \frac{\sqrt{r'}}{\sqrt{(R_2-r')(r'-R_1)}} \frac{K(k)}{m} dr'. \quad (11)$$

In order to get rid of divergences in the denominator of the integrand in (11) in the upcoming numerical calculations, we will make a replacement of the integration variable [13]:

$$r' = a_1(1 + e\sin\eta), \quad \eta = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad (12)$$

where the semimajor axis and the eccentricity of the auxiliary orbit of the star are equal

$$a_1 = \frac{R_1+R_2}{2}, \quad e = \frac{R_2-R_1}{2a_1}. \quad (13)$$

After replacement (12), the radical will disappear, and expression (11) will take the form:

$$W_{\rm mut} = -\frac{8GM_1M_2}{\pi^2\sqrt{r_2 a_1}} \int_0^{\frac{\pi}{2}} d\theta$$

$$\times \int_{-\pi/2}^{\pi/2} \frac{\sqrt{1+e\sin\eta}}{m} K(k)\, d\eta. \quad (14)$$

Here,

$$m = \sqrt{\frac{a_1(1+e\sin\eta)}{r_2} + \frac{r_2}{a_1(1+e\sin\eta)} + 2\sqrt{n}}. \quad (15)$$

The graph of the mutual energy of two rings, calculated using formula (14), is shown in Fig. 2.

Figure 3 shows the 3D dependence of the normalized mutual energy $W_{\rm mut}/\left(-\frac{GM_1M_2}{\pi^2(R_1+R_2)}\right)$ on the tilt angle α and radius of the narrow ring r_2.

In addition, Fig. 4 shows the projection of the indicated dependence of the normalized gravitational energy $W_{\rm mut}/\left(-\frac{GM_1M_2}{\pi^2(R_1+R_2)}\right)$.

3. MOMENT OF FORCES BETWEEN THE RINGS

To find the precession of the ring nodes, it is necessary to calculate the moment of gravitational forces M between the rings. Direct calculation of

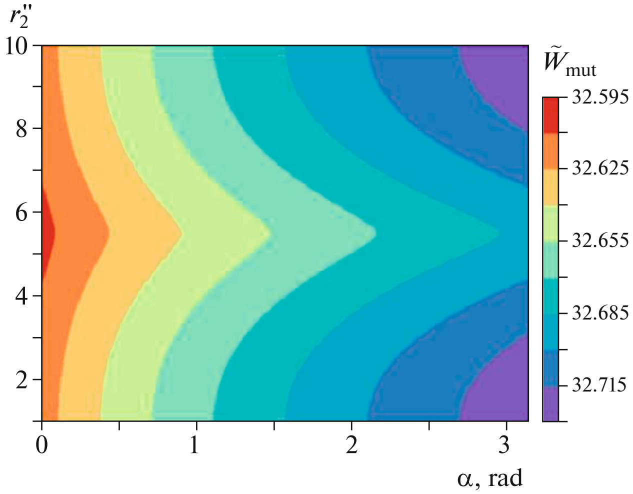

Fig. 4. Projection of the graph of the dependence of the normalized gravitational mutual energy $\tilde{W}_{\rm mut} = W_{\rm mut}/\left(-\frac{GM_1M_2}{\pi^2(R_1+R_2)}\right)$.

the moment of forces between the gravitating rings is a difficult task. However, the advantage of our approach is that, knowing the mutual energy of the two rings $W_{\rm mut}$, the component of the moment of forces along the axis Ox_1 can be expressed through the derivative of the mutual energy $W_{\rm mut}$ with respect to the angle of inclination α [14]:

$$M = \frac{\partial}{\partial\alpha}W_{\rm mut}. \tag{16}$$

The torque graph for a system of two rings of this type is shown in Fig. 5.

According to the calculations (see Fig. 5), with the inclination of the rings known from observations $\alpha \approx 62°$, the moment of force between them is equal to

$$M \approx -0.447\frac{8GM_1M_2}{\pi^2\sqrt{a_1r_2}}. \tag{17}$$

This result is used in Section 5.

4. LAPLACE PLANE FOR THE RING SYSTEM

Assuming that the motion of the stars in the disks occurs under the attraction of the black hole, it can be shown that the angular moments (moduli) are:

$$L^{(1)} = M_1\sqrt{GM_{\rm bh}\frac{2R_1R_2}{R_1+R_2}};$$
$$L^{(2)} = M_2\sqrt{GM_{\rm bh}r_2}, \tag{18}$$

where $M_{\rm bh}$ is the mass of the central black hole.

Let us introduce a special plane perpendicular to the vector $\mathbf{L}_t$ of the total orbital moment of the ring

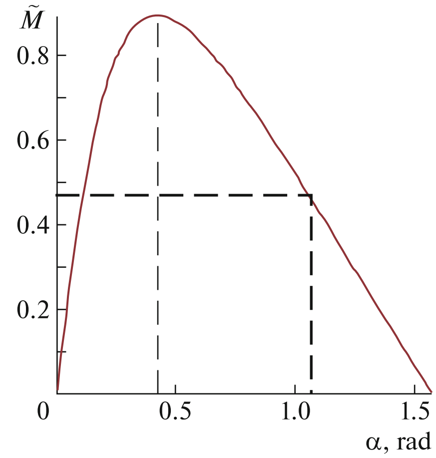

Fig. 5. Dependence of the normalized moment of forces $\tilde{M} = M/\left(-\frac{8GM_1M_2}{\pi^2\sqrt{a_1r_2}}\right)$ between the wide R-ring and the narrow ring on the angle of mutual inclination α. The curve has a maximum at $\alpha \approx 0.433\,(24.8°)$. The maximum point on this curve coincides with the inflection point of the graph in Fig. 1. The position of the system "disk 1—disk 2" ($\alpha \approx 62°$) is marked with bold strokes.

system. In celestial mechanics, this plane is usually called the Laplace plane. The peculiarity of the problem under consideration is that here the angular momentum vector $\mathbf{L}^{(1)}$ of disk 1 will be directed below the Laplace plane, and the vector $\mathbf{L}^{(2)}$ of disk 2 will be directed above this plane (see Fig. 6).

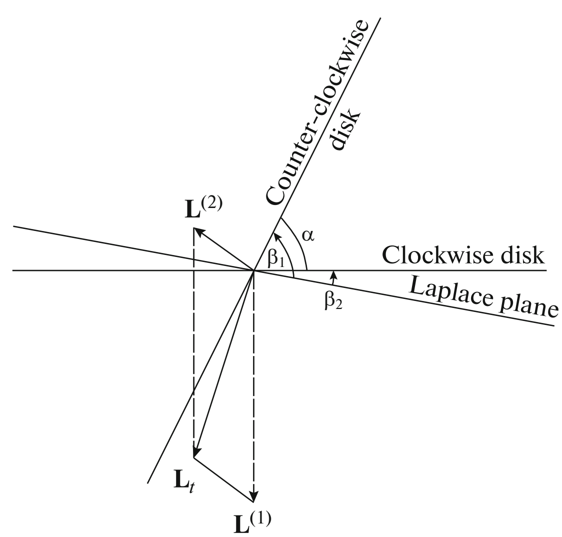

Fig. 6. Scheme of orbital angular momentum vectors $\mathbf{L}^{(1)}$ and $\mathbf{L}^{(2)}$ for two rings. The Laplace plane is shown, as well as the angle of inclination α of the rings and auxiliary angles β_1 and β_2. According to [11].

In the Cartesian coordinate system, the projections of the angular momentum vectors are equal (Fig. 6)

$$L_1^{(1)} = L^{(1)} \sin\beta_2; \quad L_1^{(2)} = -L^{(2)} \sin\beta_1;$$
$$L_2^{(1)} = L^{(1)} \cos\beta_2; \quad L_2^{(2)} = L^{(2)} \cos\beta_1, \qquad (19)$$

where β_1 and β_2 are auxiliary angles to the Laplace plane. Then the perpendicularity condition is satisfied if

$$L^{(1)} \sin\beta_2 = L^{(2)} \sin\beta_1. \qquad (20)$$

Thus, we have a system of equations for angles β_1 and β_2:

$$\frac{\sin\beta_1}{\sin\beta_2} = \frac{L^{(1)}}{L^{(2)}} = \frac{M_1}{M_2}\sqrt{\frac{2R_1R_2}{r_2\,(R_1+R_2)}} = \gamma;$$
$$\beta_1 - \beta_2 = \alpha. \qquad (21)$$

Solving system of equations (21), we find:

$$\sin\beta_1 = \frac{\gamma\sin\alpha}{\sqrt{\gamma^2 + 2\gamma\cos\alpha + 1}};$$
$$\sin\beta_2 = \frac{\sin\alpha}{\sqrt{\gamma^2 + 2\gamma\cos\alpha + 1}}. \qquad (22)$$

5. APPLICATION OF THE METHOD TO STELLAR DISKS IN THE CENTRAL PARSEC OF THE GALAXY

Let us apply the two-ring model to estimate the precession time of their planes. Recall that at the center of our Galaxy there is a supermassive black hole (SMBH) with a mass of $M_{\rm bh} = 4.3 \times 10^6 M_\odot$ [3]. This supermassive black hole is surrounded by a compact cluster of B-type stars with randomly oriented orbits [15]. Outside the nuclear cluster of stars there are two stellar disks under consideration: disk 1 and disk 2. Disk 1 has the following parameters [16, 17]:

$$R_1 \approx 1''\ (0.04\,\text{pc}), \quad R_2 \approx 10''\ (0.4\,\text{pc}),$$
$$M_1 \approx 3 \times 10^5 M_\odot. \qquad (23)$$

The thickness of the disk can be neglected here. The cavity inside disk 1 is relatively narrow ($0 < r \leq 1''$), the inner edge, according to observations, is sharp. The potential increases monotonically inside this cavity, then begins to decrease smoothly inside the disk, and beyond the outer edge of the disk the potential decreases even more steeply [7].

The second stellar ring (disk 2), is described by the parameters [3]:

$$R_1 \approx 5''\ (0.2\,\text{pc}), \quad R_2 \approx 15''\ (0.6\,\text{pc}),$$
$$M_2 \approx 5 \times 10^3 M_\odot. \qquad (24)$$

To simplify the calculations of the moment of forces and the precession of the rings, we model the second disk, whose mass is approximately 60 times less than the mass of the first disk, as a narrow ring with parameters

$$r_2 \approx 12''\ (0.48\,\text{pc}), M_2 \approx 5 \times 10^3 M_\odot. \qquad (25)$$

From observations [3], the angle of inclination of the rings is also known

$$\alpha \approx 62^\circ\ (1.082\ \text{rad})\,. \qquad (26)$$

For the known parameters of the disks (23), (24) we find, according to (18), the ratio of the angular momenta of the disks:

$$\gamma = \frac{L^{(1)}}{L^{(2)}} = \frac{M_1}{M_2}\sqrt{\frac{2R_1R_2}{r_2\,(R_1+R_2)}} \approx 23.355. \qquad (27)$$

Now, using formulae (22), taking into account (27), we calculate the auxiliary angles:

$$\beta_1 \approx 59^\circ.8717; \quad \beta_2 \approx 2^\circ.1223. \qquad (28)$$

Since the angle β_2 is small, the Laplace plane almost coincides with the plane of disk 1. This is understandable since the mass of the main disk is almost 60 times greater than the mass of the ring.

Let us now calculate the precession time of the nodes using the formula [11, 14, 18]

$$\frac{d\psi}{dt} = \frac{M}{L^{(2)}\sin\beta_1}. \qquad (29)$$

Substituting into (29) the expression for the moment of forces $\mathbf{M}$ from (17) and the angular momentum $L^{(2)}$ of the ring from (18), after reductions and transformations, we obtain the frequency of nodal precession in the form:

$$\frac{d\psi}{dt} = 0.447\frac{8GM_1}{\pi^2\sqrt{GM_{bh}a_1}\,r_2\sin\beta_1}, \qquad (30)$$

where, we recall, the semi-major a_1 of the auxiliary orbit of the star is given in (13). Then the period of precession of the nodes will be equal to

$$T_\Omega^{(2)} = \frac{2\pi}{\dot{\psi}^{(2)}} = \frac{\pi^3\sqrt{GM_{bh}a_1}\,r_2\sin\beta_1}{1.788GM_1}. \qquad (31)$$

Substituting quantities known from observations here, we ultimately obtain

$$T_\Omega^{(2)} \approx 3.53 \times 10^5\ \text{years}. \qquad (32)$$

The lines of the nodes of both rings move in the Laplace plane with the same angular velocity but in opposite directions. Indeed, due to the condition of the existence of the Laplace plane (20), the period of the nodal precession of the first disk turns out to be the same:

$$T_{\Omega}^{(1)} = T_{\Omega}^{(2)} \approx 3.53 \times 10^5 \text{ years.} \tag{33}$$

CONCLUSIONS

In this paper, a dynamic model of a system of two interacting stellar disks (rings) located in the central parsec of the Galaxy is constructed. The mass and geometric parameters of both disks are known from observations. The most massive and wide ring with retrograde motion of the star (here disk 1) is here modeled by an R-disk with a small cutout in the center. Let us recall [12, 13] that the R-disk model can be formed not only due to the precession of the apsidal lines of the star's orbit, but also due to the fact that, in agreement with observations, many stars move around black holes in orbits with apsidal lines uniformly distributed along the azimuth. Another stellar system with prograde rotation, designated here as disk 2, is represented by a weighted average narrow the circular ring inclined to the plane of the R-disk at an angle of $\alpha \approx 62^\circ$.

The mutual gravitational energy $W_{\rm mut}$ of the rings was found, and the derivative of this energy with respect to the angle α, expressing the moment of force $\mathbf{M}$ between the rings, was calculated. Graphs of these values depending on the angle of inclination α were plotted. The angular momenta of the rings $\mathbf{L}^{(1)}$ and $\mathbf{L}^{(2)}$, the ratio of the moduli of which $L^{(1)}/L^{(2)} \approx 23.36$, were also calculated. For the system of rings, the Laplace plane and the angles of its orientation were determined.

Mutual disturbance of the rotating rings leads to precession of the nodes. It has been established that in the Laplace plane the lines of the nodes of the first and second rings move with the same angular velocity, but in opposite directions. The period of nodal precession for both rings $T_\Omega \approx 3.53 \times 10^5$ years has been found. This time of precession of the nodes, on the scale of the Galaxy, turns out to be quite short. The reason is that, when replacing a thin ring (see work [11]) with a wide R-ring of the same mass, the magnitude of the moment of force increased noticeably.

It is also known from observations that the angle between the node lines of the disks is very large and equal to [3]

$$\chi \approx 243^\circ \pm 14^\circ. \tag{34}$$

The discovered rather rapid movement of the node lines in opposite directions explains why the node lines of the stellar disks are not collinear and are directed at a large angle to each other.

FUNDING

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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